Math 2135: Linear Algebra

Assignment 3

Solutions

1. **There is an interesting function $\mathbb{C} \to \mathbb{C}$ called *complex conjugation* and denoted $z \mapsto \bar{z}$. It takes a complex number $z = a + b\sqrt{-1}$, and $a, b \in \mathbb{R}$, to the *conjugate* number $\bar{z} = a - b\sqrt{-1}$.**

   (a) **Think of $V = \mathbb{C}$ as a vector space over $\mathbb{F} = \mathbb{R}$. Is complex conjugation a linear map?**

   Yes. Given complex numbers $z_1, z_2 \in V = \mathbb{C}$, we do have $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, so complex conjugation respects vector addition. Furthermore, if $\lambda \in \mathbb{F} = \mathbb{R}$, and if $z \in V = \mathbb{C}$, then $\overline{\lambda z} = \overline{\lambda}\overline{z} = \lambda\overline{z}$, so complex conjugation respects scalar multiplication.

   (b) **Think of $V = \mathbb{C}$ as a vector space over $\mathbb{F} = \mathbb{C}$. Is complex conjugation a linear map?**

   No. Let $z \in V = \mathbb{C}$ be a nonzero complex number, for example $z = 1$. Let $\lambda \in \mathbb{F} = \mathbb{C}$ be a complex number which is not purely real, for example $\lambda = \sqrt{-1}$. Then $\overline{\lambda z} = \overline{\lambda}\overline{z} \neq \lambda\overline{z}$. So complex conjugation does not respect scalar multiplication.

2. **Given any $b, c \in \mathbb{R}$, define a function $T : \mathbb{R}^3 \to \mathbb{R}^2$ by**

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

   Show that T is linear if and only if $b = c = 0$.

   Let us check whether T respects vector addition. In other words, given two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, we wish to check whether $T(v_1 + v_2) \stackrel{?}{=} T(v_1) + T(v_2)$. Spelled out, this is asking whether

$$\begin{aligned}&(2x_1 - 4y_1 + 3z_1 + b, 6x_1 + cx_1y_1z_1) + (2x_2 - 4y_2 + 3z_2 + b, 6x_2 + cx_2y_2z_2)\\ &\quad \stackrel{?}{=} (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b, 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2))\end{aligned}$$

   is always an equality of vectors in $\mathbb{R}^2$. Matching the coordinates, we are asking:

$$2x_1 - 4y_1 + 3z_1 + b + 2x_2 - 4y_2 + 3z_2 + b \stackrel{?}{=} 2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b$$
$$6x_1 + cx_1y_1z_1 + 6x_2 + cx_2y_2z_2 \stackrel{?}{=} 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2)$$

   After some light rearranging, we see that the first equation holds if and only if

$$2b \stackrel{?}{=} b$$

   and the second holds if and only of, for every $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$, we have

$$c(x_1y_1z_1 + x_2y_2z_2) \stackrel{?}{=} c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2).$$

The first equation holds if and only if $b = 0$. The second holds if and only if $c = 0$: the if direction is obvious, and the only if direction is readily seen by setting $x_1 = \cdots = z_2 = 1$.

We have thus shown that T respects vector addition if and only if $b = c = 0$. It remains to show that if $b = c = 0$, then T respects scalar multiplication. But, letting $\lambda \in \mathbb{R}$, and taking $b = c = 0$, we compute:

$$T\lambda(x, y, z) = T(\lambda x, \lambda y, \lambda z) = (2\lambda x - 4\lambda y + 3\lambda z, 6\lambda x) = \lambda(2x - 4y + 3z, 6x) = \lambda T(x, y, z).$$

3. **Let $V \subset \mathbb{R}^{\mathbb{R}}$ be the vector space of all differentiable functions. Given $f \in V$, denote its derivative by f'.**

   (a) **Is the function $V \to \mathbb{R}^2$ that sends $f \mapsto \left(f(1) + f'(2), \int_0^3 f(x)\,\mathrm{d}x\right)$ linear? Why or why not?**

   $\boxed{\text{Yes}}$, this function is linear. To see this, it is helpful to inspect the pieces of the function. First, for any real number $r \in \mathbb{R}$, the map $V \to \mathbb{R}$ sending $f \mapsto f(r)$ is linear. Indeed, the function $f + g$ is defined so that $(f + g)(r) = f(r) + g(r)$ for every r, and for $\lambda \in \mathbb{R}$ the function λf is defined so that $(\lambda f)(r) = \lambda f(r)$ for every r. So for example $f \mapsto f(1)$ is a linear transformation.

   Second, taking derivatives is linear. Indeed, $(f+g)' = f'+g'$ and $(\lambda f)' = \lambda' f + \lambda f' = \lambda f'$ since $\lambda \in \mathbb{R}$ is constant (so $\lambda' = 0$). Since evaluating is also linear, we see that $f \mapsto f'(2)$ is linear.

   Third, definite integrals are linear: for fixed real numbers $a, b \in \mathbb{R}$, and for any integrable functions f, g, we know that $\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$, and for any integrable function f and any real number λ, we know that $\int_a^b \lambda f(x)\,\mathrm{d}x = \lambda \int_a^b f(x)\,\mathrm{d}x$. (All differential functions are continuous, and all continuous functions are integrable.)

   Sums of linear functions are linear. And the addition and scalar multiplication in $\mathbb{R}^2$ are defined component wise: this implies that if you have a transformation valued in $\mathbb{R}^2$ such that each component is linear, then the whole transformation is linear.

   More directly, what we are saying is: For any differentiable functions f, g and any constant λ,

$$\left((f + g)(1) + (f + g)'(2), \int_0^3 (f + g)(x)\,\mathrm{d}x\right) = \left(f(1) + f'(2), \int_0^3 f(x)\,\mathrm{d}x\right) + \left(g(1) + g'(2), \int_0^3 g(x)\,\mathrm{d}x\right),$$
$$\left(\lambda f(1) + \lambda f'(2), \int_0^3 \lambda f(x)\,\mathrm{d}x\right) = \lambda\left(f(1) + f'(2), \int_0^3 f(x)\,\mathrm{d}x\right).$$

   (b) **Is the function $V \to \mathbb{R}$ that sends $f \mapsto \int_0^3 x^2 f(x)\,\mathrm{d}x$ linear? Why or why not?**

   $\boxed{\text{Yes}}$, this function is linear. We have already remarked that the assignment $f \mapsto \int_0^3 f(x)\,\mathrm{d}x$ is linear. The question asks about what you get if you precompose this operation with the assignment $f(x) \mapsto x^2 f(x)$. In general, fix any differentiable function $h(x)$. Then the function $V \to V$ which sends $f \mapsto hf$ is linear:

$$h(x)\,(f(x) + g(x)) = h(x)\,f(x) + h(x)\,g(x), \quad h(x)\,\lambda f(x) = \lambda h(x)\,f(x), \quad \forall f, g \in V,\ \lambda \in \mathbb{R}$$

   More directly, what we are saying is: For any differentiable functions f, g and any constant λ,

$$\int_0^3 x^2(f(x)+g(x))\,\mathrm{d}x = \int_0^3 \big(x^2 f(x) + x^2 g(x)\big),\mathrm{d}x = \int_0^3 x^2 f(x)\,\mathrm{d}x + \int_0^3 x^2 g(x)\,\mathrm{d}x,$$
$$\int_0^3 x^2 \lambda f(x)\,\mathrm{d}x = \int_0^3 \lambda x^2 f(x)\,\mathrm{d}x = \lambda \int_0^3 x^2 f(x)\,\mathrm{d}x.$$

(c) **Is the function $V \to \mathbb{R}$ that sends $f \mapsto f'(2)^2$ linear? Why or why not?**

$\boxed{\text{No}}$, this function is not linear. Let us write T for this operation, i.e. $T(f) = f'(2)^2$. Now consider the case $f(x) = g(x) = x$. Then $f'(2) = g'(2) = 1$, and so

$$T(f) + T(g) = 1 + 1 = 2.$$

On the other hand, $(f+g)(x) = 2x$, and so

$$T(f+g) = T(2x) = 2^2 = 4 \neq 2.$$

Thus $T(f+g) \neq T(f) + T(g)$ for some $f, g \in V$, and so T does not respect vector addition.

T also does not respect scalar multiplication, which can be seen for example by taking $\lambda = 2$ and $f(x) = x$.

Remark: There do exist functions f, g such that $T(f+g) = T(f) + T(g)$, for example $f = g = 0$. There also exist numbers λ such that $T(\lambda f) = \lambda T(f)$, for example $\lambda = 0$ or $\lambda = 1$. However, for *most* functions f, g and for most numbers λ, you will find that $T(f+g) \neq T(f) + T(g)$ and $T(\lambda g) \neq \lambda T(f)$.

4. **Suppose $T : \mathbb{F}^4 \to \mathbb{F}^2$ is a linear map such that**

$$\ker(T) = \{(x_1, x_2, x_3, x_4) \textbf{ s.t. } x_1 = 5x_2 \textbf{ and } x_3 = x_1 + x_4\}.$$

Show that T is surjective.

We know that $\dim \mathbb{F}^4 = 4$ and $\dim \mathbb{F}^2 = 2$. We also know that

$$\dim \mathbb{F}^4 = \dim \ker T + \dim \operatorname{im} T.$$

Thus, if we can show that $\dim \ker T = 2$, then we will know that $\dim \operatorname{im} T = 2$. But $\operatorname{im} T \subseteq \mathbb{F}^2$, and so if $\dim \operatorname{im} T = 2$, then we must have equality $\operatorname{im} T = \mathbb{F}^2$.

It thus remains to understand $\ker T$. We claim that $v_1 := (5,1,5,0)$ and $v_2 := (0,0,1,1)$ are together a basis for $\ker T$. They are obviously linearly independent, and obviously both in $\ker T$. So the only thing to check is whether they span $\ker T$. Well, suppose that $v = (x_1, x_2, x_3, x_4) \in \ker T$. Then $x_1 = 5x_2$ and $x_3 = 5x_2 + x_4$. Set $\alpha_1 := x_2$ and $\alpha_2 := x_4$. Then

$$\begin{aligned}\alpha_1 v_1 + \alpha_2 v_2 = x_2(5,1,5,0) + x_4(0,0,1,1) &= (5x_2, x_2, 5x_2, 0) + (0,0,x_4,x_4) \\ &= (5x_2, x_2, 5x_2 + x_4, x_4) = (x_1, x_2, x_3, x_4) = v.\end{aligned}$$

So these two vectors span, and hence are a basis for, $\ker T$. So $\dim \ker T = 2$, which is what we wanted to prove.

5. (a) **Find a linear map $T : \mathbb{F}^2 \to \mathbb{F}^2$ such that $\ker(T) = \operatorname{im}(T)$ or show that one does not exist.**

For example,

$$T(x, y) := (y, 0)$$

has $\ker T = \{(x, y) : y = 0\} = \{(x, 0)\}$, where x is arbitrary, whereas $\operatorname{im} T = \{(y, 0)\}$ where y is arbitrary. So $\ker T = \operatorname{im} T$.

(b) **Find a linear map $T : \mathbb{F}^3 \to \mathbb{F}^3$ such that $\ker(T) = \operatorname{im}(T)$ or show that one does not exist.**

There does not exist such a map. Indeed, for any map $T : \mathbb{F}^3 \to \mathbb{F}^3$, we definitely have

$$3 = \dim \mathbb{F}^3 = \dim \ker T + \dim \operatorname{im} T.$$

Suppose that $\ker T = \operatorname{im} T$ and that $n = \dim \ker T = \dim \operatorname{im} T$. Then we would have $3 = 2n$, i.e. $n = \frac{3}{2}$. But dimensions are always integers.

6. **Let $S : U \to V$ and $T : V \to W$ be linear maps. Show that if any two of the three maps S, T, and their composition $TS : U \to W$, are invertible, then so is the third one.**

Suppose first that S and T are invertible, with inverses S^{-1} and T^{-1} respectively. In other words, $SS^{-1} = I_V$, $S^{-1}S = I_U$, $TT^{-1} = I_W$, and $T^{-1}T = I_V$. Set $X := S^{-1}T^{-1}$. We claim that X is an inverse to TS. In other words, we claim that $XTS = I_U$ and $TSX = I_W$. We now check these:

$$S^{-1}T^{-1}TS = S^{-1}I_V S = S^{-1}S = I_U, \qquad TSS^{-1}T^{-1} = TI_V T^{-1} = I_W.$$

These equations confirm that $X = S^{-1}T^{-1}$ is indeed an inverse to TS

Next, suppose that S and TS are invertible, with inverses S^{-1} and $(TS)^{-1}$ respectively. In other words, $SS^{-1} = I_V$, $S^{-1}S = I_U$, $(TS)^{-1}TS = I_U$, and $TS(TS)^{-1} = I_W$. Set $X := S(TS)^{-1}$. We wish to claim that X is an inverse to T, i.e. that $TX = I_W$ and $XT = I_V$. One of these is easy:

$$TX = TS(TS)^{-1} = I_W.$$

To show that X is an inverse to T, we must also show that $XT = I_V$. This is harder because $XT = S(TS)^{-1}T$. If we could move the S over next to the T, then we'd be able to use the equation $(TS)^{-1}TS = I_U$. Well, let's remember that we have access to S^{-1} such that $SS^{-1} = I_V$. Then:

$$XT = S(TS)^{-1}T = S(TS)^{-1}TI_V = S(TS)^{-1}TSS^{-1} = SI_U S^{-1}SS^{-1} = I_V.$$

So indeed X is an inverse to T.

The third case is almost the same as the second case. Suppose that T and TS are invertible, with T^{-1} and $(TS)^{-1}$ respectively. We claim that $X := (TS)^{-1}T$ is an inverse to S. To check this, we compute:

$$XS = (TS)^{-1}TS = I_U,$$
$$SX = S(TS)^{-1}T = I_V S(TS)^{-1}T = T^{-1}TS(TS)^{-1}T = T^{-1}I_W T = T^{-1}T = I_V.$$