Math 2135: Linear Algebra

Assignment 3

due 9 February 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please indicate (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. There is an interesting function $\mathbb{C} \to \mathbb{C}$ called *complex conjugation* and denoted $z \mapsto \bar{z}$. It takes a complex number $z = a + b\sqrt{-1}$, and $a, b \in \mathbb{R}$, to the *conjugate* number $\bar{z} = a - b\sqrt{-1}$.
   (a) Think of $V = \mathbb{C}$ as a vector space over $\mathbb{F} = \mathbb{R}$. Is complex conjugation a linear map?
   (b) Think of $V = \mathbb{C}$ as a vector space over $\mathbb{F} = \mathbb{C}$. Is complex conjugation a linear map?

2. Given any $b, c \in \mathbb{R}$, define a function $T : \mathbb{R}^3 \to \mathbb{R}^2$ by
$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$
   Show that T is linear if and only if $b = c = 0$.

3. Let $V \subset \mathbb{R}^{\mathbb{R}}$ be the vector space of all differentiable functions. Given $f \in V$, denote its derivative by f'.
   (a) Is the function $V \to \mathbb{R}^2$ that sends $f \mapsto \left(f(1) + f'(2), \int_0^3 f(x)\,\mathrm{d}x\right)$ linear? Why or why not?
   (b) Is the function $V \to \mathbb{R}$ that sends $f \mapsto \int_0^3 x^2 f(x)\,\mathrm{d}x$ linear? Why or why not?
   (c) Is the function $V \to \mathbb{R}$ that sends $f \mapsto f'(2)^2$ linear? Why or why not?

4. Suppose $T : \mathbb{F}^4 \to \mathbb{F}^2$ is a linear map such that
$$\ker(T) = \{(x_1, x_2, x_3, x_4) \text{ s.t. } x_1 = 5x_2 \text{ and } x_3 = x_1 + x_4\}.$$
   Show that T is surjective.

   Note: As in lecture, $\ker(-)$ and $\mathrm{null}(-)$ mean the same thing. Similarly, $\mathrm{im}(-)$ and $\mathrm{range}(-)$ mean the same thing.

5. (a) Find a linear map $T : \mathbb{F}^2 \to \mathbb{F}^2$ such that $\ker(T) = \mathrm{im}(T)$ or show that one does not exist.
   (b) Find a linear map $T : \mathbb{F}^3 \to \mathbb{F}^3$ such that $\ker(T) = \mathrm{im}(T)$ or show that one does not exist.

6. Let $S : U \to V$ and $T : V \to W$ be linear maps. Show that if any two of the three maps S, T, and their composition $TS : U \to W$, are invertible, then so is the third one.

   Remark: This result is called the *two-out-of-three theorem for invertibility.*