Math 2135: Linear Algebra

Assignment 4

Solutions

1. **Let V and W be vector spaces, and let $T : V \to W$ be any function. The *graph* of T is defined to be the subset $\mathrm{graph}(T) \subset V \times W$ defined by**

$$\mathrm{graph}(T) := \{(v, Tv) \textbf{ s.t. } v \in V\}.$$

   Prove that T is linear if and only if $\mathrm{graph}(T)$ is a vector subspace of $V \times W$.

   We first remark that $(0, T0) \in \mathrm{graph}(T)$, and so $\mathrm{graph}(T)$ is nonempty.

   We will use the fact that the function $\mathrm{graph}(T) \to V$ given by sending $(v, Tv) \mapsto v$ is a bijection. In other words:

$$\boxed{(v, w) \in \mathrm{graph}(T) \text{ if and only if } w = Tv.}$$

   Suppose that (v_1, Tv_1) and (v_2, Tv_2) are two points in $\mathrm{graph}(T)$. Then their sum is $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2)$. Taking $v := v_1 + v_2$ and $w := Tv_1 + Tv_2$ in the boxed statement above, we see that this sum (v, w) is in $\mathrm{graph}(T)$ if and only if $w = Tv_1 + Tv_2 = Tv = T(v_1 + v_2)$. Asking this to hold for all v_1, v_2, we thus find that $\mathrm{graph}(T)$ is closed under addition if and only if T is additive.

   Now suppose that (v, Tv) is some point in $\mathrm{graph}(T)$, and $\lambda \in \mathbb{F}$ is some scalar. Then $\lambda(v, Tv) = (\lambda v, \lambda Tv)$. Again applying the boxed statement, we see that this point is in $\mathrm{graph}(T)$ if and only if $\lambda Tv = T\lambda v$. Asking this to hold for all v, λ, we thus find that $\mathrm{graph}(T)$ is closed under scalar multiplication if and only if T is homogeneous.

   Together, we see that the two conditions (additivity and homogeneity) for T to be linear match the two conditions (closure under addition and scalar multiplication) for the nonempty(!) set $\mathrm{graph}(T)$ to be a vector subspace of $V \times W$.

2. **Let V_1, V_2, W be vector spaces.**

   (a) **Find an isomorphism between $\mathcal{L}(W, V_1 \times V_2)$ and $\mathcal{L}(W, V_1) \times \mathcal{L}(W, V_2)$.**

   Given any function $T : W \to V_1 \times V_2$, linear or not, define $T_1 : W \to V_1$ to be the function whose value on $w \in W$ is the first coordinate of $T(w)$, and let $T_2 : W \to V_2$ be the function whose value is the second coordinate. In other words, T_1, T_2 are defined by the formula

$$\boxed{T(w) = (T_1(w), T_2(w)).}$$

   In the other direction, given any functions $T_1 : W \to V_1$ and $T_2 : W \to V_2$, linear or not, the same formula defines a function $T : W \to V_1 \times V_2$.

   We claim that if T, T_1, and T_2 are related in this way, then T is linear if and only if T_1 and T_2 are both linear. This claim follows simply from the way that vector addition and scalar multiplication are defined in $V_1 \times V_2$. Indeed, given $w, w' \in W$ and $\lambda \in \mathbb{F}$, we see that

$$T(\lambda w + w') = (T_1(\lambda w + w'), T_2(\lambda w + w')), \qquad \lambda T(w) + T(w') = (\lambda T_1(w) + T_1(w'), \lambda T_2(w) + T_2(w')),$$

but linearity of T is the statement that the above left-hand sides are equal for all w, w', whereas the statement that the above right-hand sides are equal for all w, w' is precisely linearity of both T_1 and T_2. Another way to see this is to note that T_1, T_2 are nothing but the compositions of T with the projection maps $\pi_1 : V_1 \times V_2 \to V_1$ and $\pi_2 : V_1 \times V_2$ defined by $\pi_i(v_1, v_2) = v_i$, and that these projection maps are linear, and that the composition of linear maps is linear.

Thus we see that the boxed formula above, read in its two directions, defines a bijective correspondence between $\mathcal{L}(W, V_1 \times V_2)$ and $\mathcal{L}(W, V_1) \times \mathcal{L}(W, V_2)$. It remains to check that this correspondence is linear. In other words, given $T = (T_1, T_2)$ and $T' = (T_1', T_2')$ and $\lambda \in \mathbb{F}$, we must check that, for every $w \in W$,

$$(\lambda T + T')(w) = ((\lambda T_1 + T_1')(w), (\lambda T_2 + T_2')(w)).$$

This follows from unpacking — for example, the left-hand side is $\lambda Tw + T'w$ — and using the definition of the vector space structure on $V_1 \times V_2$.

(b) **Find an isomorphism between $\mathcal{L}(V_1 \times V_2, W)$ and $\mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W)$.**

Note that $(v_1, v_2) = (v_1, 0) + (0, v_2)$ as vectors in $V_1 \times V_2$, for any vectors $v_1 \in V_1$ and $v_2 \in V_2$. Now suppose that $T : V_1 \times V_2 \to W$ is any linear map. Then

$$T(v_1, v_2) = T(v_1, 0) + T(0, v_2).$$

Define functions $T_1 : V_1 \to W$ and $T_2 : V_2 \to W$ by $T_1(v_1) = T(v_1, 0)$ and $T_2(v_2) = T(0, v_2)$. Then the displayed equation equation is equivalent to

$$\boxed{T(v_1, v_2) = T_1(v_1) + T_2(v_2).}$$

If T is linear, then so are T_1, T_2, since these are nothing but the compositions of T with the inclusions $\iota_1 : V_1 \to V_1 \times V_2$ and $\iota_2 : V_2 \to V_1 \times V_2$ defined by $\iota_1(v_1) = (v_1, 0)$ and $\iota_2(v_2) = (0, v_2)$. On the other hand, suppose that $T_1 : V_1 \to W$ and $T_2 : V_2 \to W$ are any linear maps, and define a function $T : V_1 \times V_2 \to W$ by the boxed equation. Then T will be linear:

$$\begin{aligned} T(\lambda(v_1, v_2) + (v_1', v_2')) = T(\lambda v_1 + v_1', \lambda v_2 + v_2') &= T_1(\lambda v_1 + v_1') + T_2(\lambda v_2 + v_2') \\ &= \lambda T_1 v_1 + T_1 v_1' + \lambda T_2 v_2 + T_2 v_2' = \lambda T(v_1, v_2) + T(v_1', v_2') \end{aligned}$$

Thus we see that the boxed equation defines a bijection between $\mathcal{L}(V_1 \times V_2, W)$ and $\mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W)$. It is straightforward to show that this bijection is linear. Indeed, one must show that if T, T_1, T_2 are as in the boxed equation, and if T', T_1', T_2' are, then for any λ,

$$(\lambda T + T')(v_1, v_2) = (\lambda T_1 + T_1')(v_1) + (\lambda T_2 + T_2')(v_2)$$

which can be checked by expanding both sides.

3. **Let V be a vector space over $\mathbb{F}$. Prove that a nonempty subset $X \subset V$ is an affine subspace (for some vector subspace $U \subset V$) if and only if, for any $v, w \in X$ and for any $\lambda \in \mathbb{F}$, $\lambda v + (1-\lambda)w \in X$.**

   Note that the statement $\lambda v + (1-\lambda)w \in X$ can be rewritten as

   $$w + \lambda(v - w) \in X.$$

   If X is an affine subset for U, then for any $v, w \in X$ and for any $\lambda \in \mathbb{F}$, we have $v - w \in U$ and so $\lambda(v-w) \in U$ and so $w + \lambda(v-w) \in X$. This proves the "only if" direction. In other words, we have shown that if X is an affine subspace, then $\lambda v + (1-\lambda)w \in X$ for all $v, w \in X$.

   For the "if" direction (i.e. if $\lambda v + (1-\lambda)w \in X$ for all $v, w \in X$, then X is an affine subspace), we use that X is nonempty to fix a basepoint $x \in X$. Set $U := X - x$ to be the set of vectors of the form $v - x$ where $v \in X$. Then $X = x + U$, and it suffices to show that U is a vector subspace. Note that $0 = x - x \in U$.

   To this end, we first argue that U is closed under scalar multiplication. Let $u = v - x \in U$. We want to show that $\lambda u = \lambda(v - x) = \lambda v - \lambda x$ is also in U. This is equivalent to showing that $\lambda v - \lambda x + x \in X$, which follows (by setting $w = x$) from the assumption about X.

   The only thing remaining to prove is that U is closed under vector addition. In other words, we assume that we have $u_1, u_2 \in U$ and we want to show that $u_1 + u_2 \in U$. But U is closed under scalar multiplication, so $u_1 + u_2 \in U$ if and only if $\frac{1}{2}(u_1 + u_2) \in U$. It is this latter statement that we will prove. By definition of U, we know that $u_1 + x$ and $u_2 + x$ are both in X, and our goal is to prove that $\frac{1}{2}(u_1 + u_2) + x \in X$. But taking $v = u_1 + x$ and $w = u_2 + x$ and $\lambda = \frac{1}{2}$ in the assumed property of X gives exactly this goal.

4. **Let $U_1, U_2 \subset V$ be vector subspaces. Suppose that $X_1 \subset V$ is an affine subspace for U_1 and that $X_2 \subset V$ is an affine subspace for U_2. Prove that $X_1 \cap X_2$ is either empty or an affine subspace for $U_1 \cap U_2$.**

   In other words, we wish to show the following two facts:

   (a) If $x \in X_1 \cap X_2$ and $u \in U_1 \cap U_2$, then $x + u \in X_1 \cap X_2$.
   (b) If $x, y \in X_1 \cap X_2$, then $x - y \in U_1 \cap U_2$.

   But if $x \in X_1 \cap X_2$ and $u \in U_1 \cap U_2$, then certainly $x \in X_1$ and $u \in U_1$, and since X_1 is affine for U_1, then certainly $x + u \in X_1$. By the same token, certainly $x \in X_2$ and $u \in U_2$ and so $x + u \in U_2$. But if $x + u$ is in both U_1 and U_2, then it is in $U_1 \cap U_2$. This establishes fact 1.

   Fact 2 is similar. If $x, y \in X_1 \cap X_2$, then certainly they are both in X_1, and so their difference $x - y$ is in U_1. But also certainly they are both in X_2, and so their difference is in U_2. But if $x - y$ is in both U_1 and U_2, then it is in $U_1 \cap U_2$.

5. **Let $U \subset V$ be a vector subspace, let $\iota : U \to V$ denote the "identity" map $\iota(u) = u$, and let $\pi : V \to V/U$ the quotient map. Let W be another vector space, and consider the maps**

   $$\circ\iota : \mathcal{L}(V, W) \to \mathcal{L}(U, W),\ T \mapsto T \circ \iota,$$
   $$\circ\pi : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W),\ S \mapsto S \circ \pi.$$

   Finally, let $X \subset \mathcal{L}(V, W)$ denote the subset

   $$X := \{T \in \mathcal{L}(V, W) \textbf{ s.t. } \ker(T) \supseteq U\}.$$

(a) **Show that $\circ\iota$ and $\circ\pi$ are linear, and that X is a vector subspace.**

For the linearity, we will prove the following more general statement. Let $S : U \to V$ denote any linear map. Then $\circ S : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ is linear. This statement, applied to $S = \iota$, gives the first linearity, and applied to $S = \pi$ (with V replaced by V/U and U replaced by V) gives the second linearity.

To this end, we first show that $\circ S$ is additive. Let $T_1, T_2 \in \mathcal{L}(V, W)$. Then $T_1 + T_2$ is the function which sends $v \mapsto (T_1 + T_2)(v) := T_1 v + T_2 v$. Its composition with S is thus the function which sends $u \mapsto T_1 S u + T_2 S u$. But this is precisely the function $T_1 S + T_2 S$. This confirms that $\circ S$ is additive. Homogeneity is confirmed similarly: let $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Then λT is the function $v \mapsto \lambda T v$, and so $(\lambda T) \circ S$ is the function $u \mapsto \lambda T S u$, which is precisely the function $\lambda(T \circ S)$.

We now show that X is a vector space. To do so we unpack the condition "$\ker(T) \supseteq U$": it is equivalent to saying

$$\boxed{Tu = 0 \quad \forall u \in U.}$$

Now, X is nonempty becuase the zero map $T = 0$ satisfies this condition. Suppose that $T_1, T_2 \in X$. In other words, $T_1 u = T_2 u = 0$ for all $u \in U$. But then $(T_1 + T_2)u = T_1 u + T_2 u = 0 + 0 = 0$, so $T_1 + T_2 \in X$. And if $T \in X$ and $\lambda \in \mathbb{F}$, then $(\lambda T)u = \lambda T u = \lambda 0 = 0$ for all $u \in U$, so $\lambda T \in X$.

(b) **Show that $\circ\pi$ is injective.**

A linear map is injective if and only if its kernel is zero. So the question is equivalent to showing that $\ker(\circ\pi) = \{0\}$. Unpacked, this is equivalent to showing that if $S : V/U \to W$ satisfies $S \circ \pi = 0 : V \to W$, then $S = 0$. Well, π is the function that sends $v \mapsto v + U$, and so $S \circ \pi$ is the function which sends $v \in V$ to $S(v + U)$. So saying that this is identically the zero function is saying that $S(v + U) = 0$ for every $v + U \in V/U$. But every element of V/U is of the form $v + U$, and so S vanishes identically, i.e. it is the zero function.

(c) **Show that $\mathrm{im}(\circ\pi) = X$. Conclude that $\mathcal{L}(V/U, W) \cong X \subset \mathcal{L}(V, W)$.**

We wish to show that $T \in X \subset \mathcal{L}(V, W)$ if and only if there exists an $S \in \mathcal{L}(V/U, W)$ such that $S \circ \pi = T$. Unpacked, we wish to show that the following conditions on $T : V \to W$ are equivalent:

i. $Tu = 0$ for all $u \in U$.

ii. There exists $S : V/U \to W$ such that $Tv = S(v + U)$ for all $v \in V$.

Well, if ii. holds, then $Tu = S(u + U) = S(0 + U) = T0 = 0$, so i. holds.

On the other hand, suppose that i. holds. Then recall that there is a linear map $\bar{T} : V/U \to W$ defined by the formula $\bar{T}(v + U) = Tv$. Indeed, for this formula to define a function, we need to know that if $v + U = v' + U$, then $Tv = Tv'$ (so that the formula really does return a well-defined output), but this is equivalent to the statement that $T(v - v') = 0$, which follows from the fact that $v - v' \in U$ and that T vanishes on U. Now take $S := \bar{T}$; then ii. holds.

The conclusion $\mathcal{L}(V/U, W) \cong X \subset \mathcal{L}(V, W)$ follows from the fact that $\circ\pi$ is injective by (b), and so it provides an isomorphism between its domain $\mathcal{L}(V/U, W)$ and its image X.

(d) **Show that $\ker(\circ\iota) = X$.**

By definition, $\ker(\circ\iota) \subset \mathcal{L}(V, W)$ is the set of $T : V \to W$ such that $T \circ \iota = 0$, or equivalently such that $0 = T\iota u = Tu$ for all $u \in U$. In other words, $u \in \ker T$ if $u \in U$, or in other words $U \subseteq \ker T$. So this is simply the definition of X.

(e) **Assuming that V is finite-dimensional, show that $\circ\iota$ is surjective. (In fact, $\circ\iota$ is surjective even if V is infinite-dimensional, but proving this requires some set-theoretic results that go beyond the class.) Conclude that $\mathcal{L}(U, W) \cong \mathcal{L}(V, W)/X$.**

We must show that if $S : U \to W$ is any linear map, then $S = T \circ \iota$ for some linear map $T : V \to W$. In other words, we are given the values of T on U and must extend it to all of V. Using finite-dimensionality, let's assume that $\dim V = n$ and $\dim U = m \le n$. Pick a basis $v_1, \dots, v_m$ for U, and extend it to a basis $v_1, \dots, v_m, v_{m+1}, \dots, v_n$ for V. Now define

$$Tv_i := \begin{cases} Sv_i, & i \le m, \\ 0, & i > m. \end{cases}$$

Remember that for any basis of V, any linear map $V \to W$ is uniquely determined by its values on that basis. Thus there is a unique linear map $T : V \to W$ which takes these values. It is the following: for any $v \in V$, write $v = \sum_{i=1}^n \alpha_i v_i$, and then $Tv = \sum_{i=1}^n \alpha_i T v_i$.

We claim that $T \circ \iota = S$, or in other words that $Tu = Su$ if $u \in U$. Well, suppose that $u \in U$, and write $u = \sum_{i=1}^n \alpha_i v_i$. Since $v_1, \dots, v_n$ is a basis for V, there is a unique such expression for v. Because $v_1, \dots, v_m$ is a basis for U, we can write u as a linear combination of just the first m basis vectors, i.e. we don't need the last $n - m$. Since the sum $u = \sum_{i=1}^n \alpha_i v_i$ uniquely determines the α_i, we conclude that $\alpha_{m+1} = \cdots = \alpha_n = 0$. Thus $Tv = \sum_{i=1}^n \alpha_i T v_i = \sum_{i=1}^m \alpha_i T v_i + \sum_{i=m+1}^n \alpha_i T v_i = \sum_{i=1}^m \alpha_i T v_i + \sum_{i=m+1}^n 0 T v_i = \sum_{i=1}^m \alpha_i T v_i + 0 = Su$.

This establishes the surjectivity of $\circ\iota : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$. The conclusion follows because, by (d), $\ker(\circ\iota) = X$.

To summarize these results: $\mathcal{L}(-, W)$ takes subs to quotients and quotients to subs.

6. **Let V_1, V_2, W be vector spaces. A function $T : V_1 \times V_2 \to W$ is called *bilinear* if for any $v_1 \in V_1$, the function $T(v_1, -) : V_2 \to W$ defined by $v_2 \mapsto f(v_1, v_2)$ is linear, and also for any $v_2 \in V_2$ the function $T(-, v_2) : V_1 \to W$ defined by $v_1 \mapsto T(v_1, v_2)$ is linear. Let $\mathcal{BL}(V_1, V_2; W)$ denote the set of bilinear functions $V_1 \times V_2 \to W$.**

(a) **When $V_1 = V_2 = W = \mathbb{R}$, show that the addition function $A(v_1, v_2) := v_1 + v_2$ is linear but not bilinear, whereas the multiplication function $M(v_1, v_2) := v_1 v_2$ is bilinear but not linear.**

The statement that A is linear unpacks to:

additivity $A((v_1, v_2) + (v_1', v_2')) = A(v_1, v_2) + A(v_1', v_2')$. But the left-hand side is $A(v_1 + v_1', v_2 + v_2') = v_1 + v_1' + v_2 + v_2'$ and the right-hand side is $v_1 + v_2 + v_1' + v_2'$, which agree by commutativity of addition.

homogeneity $A(\lambda(v_1, v_2)) = \lambda A(v_1, v_2)$. But the left-hand side if $A(\lambda v_1, \lambda v_2) = \lambda v_1 + \lambda v_2$, and the right-hand side is $\lambda(v_1 + v_2)$, which agree by the distributivity law.

A is not bilinear. If it were, then for any choice of v_1, the function $A(v_1, -)$ would be linear. So let's choose $v_1 = 2$ to test this. Then we are asking whether $A(2, -) : v \mapsto 2 + v$ is a linear map. It isn't, for example it is inhomogeneous ($A(2, \lambda v) = 2 + \lambda v \ne \lambda(2 + v)$).

The statement that M is bilinear unpacks to two statements. First, we are asserting that for fixed v_1, the function $M(v_1, -)$ is linear, and second we are asserting that for fixed

v_2, the function $M(-, v_2)$ is linear. Note that $V_1 = V_2 = \mathbb{R}$, and so we we will change letters: we are asserting that for $v_1 = \lambda \in \mathbb{R}$ fixed, the function $M(\lambda, -) : v \mapsto \lambda v$ is linear, and we are asserting that for $v_2 = \lambda \in \mathbb{R}$, the function $M(-, \lambda) : v \mapsto v\lambda$ is linear. These are the same function, and it is linear by the distributivity and associativity of multiplication.

(b) **Show that $\mathcal{BL}(V_1, V_2; W)$ is a vector subspace of the set $W^{V_1 \times V_2}$ of all functions.**

To see that $\mathcal{BL}(V_1, V_2; W)$ is nonempty, note that the zero map $0(-, -)$ is bilinear. Indeed, if you fix v_1, then $0(v_1, -)$ is the zero map, which is linear, and if you fix v_2, then $0(-, v_2)$ is the zero map, which is linear.

To see that $\mathcal{BL}(V_1, V_2; W)$ is closed under addition, suppose that T, T' are both bilinear. Then for any $(v_1, v_2) \in V_1 \times V_2$, we have $(T + T')(v_1, v_2) = T(v_1, v_2) + T'(v_1, v_2)$. Now, if we fix v_1, we are looking at the function $(T+T')(v_1, -) = T(v_1, -) + T'(v_1, -) : V_2 \to W$, which is a sum of two linear functions and hence linear. Similarly, if we fix v_2, then $(T + T')(-, v_2) = T(-, v_2) + T'(-, v_2) : V_1 \to W$ is a sum of two linear functions and hence linear. So $T + T'$ is bilinear.

To see that $\mathcal{BL}(V_1, V_2; W)$ is closed under scalar multiplication, suppose that T is bilinear and $\lambda \in \mathbb{F}$. Then λT is the function defined by $(\lambda T)(v_1, v_2) = \lambda\, T(v_1, v_2)$. If you fix v_1, you get the function $\lambda T(v_1, -) : V_2 \to W$, which is λ times the linear function $T(v_1, -) : V_2 \to W$, and hence linear. Similarly, if you fix v_2, you get $\lambda T(-, v_2) : V_1 \to W$, which is a multiple of a linear function and hence linear. So λT is bilinear.

(c) **Show that $\mathcal{BL}(V_1, V_2; W)$ is isomorphic to $\mathcal{L}(V_1, \mathcal{L}(V_2, W))$ and also to $\mathcal{L}(V_2, \mathcal{L}(V_1, W))$.**

There is a manifest isomorphism $\mathcal{BL}(V_1, V_2; W) \cong \mathcal{BL}(V_2, V_1; W)$ given by sending $T : V_1 \times V_2 \to W$ to the function $T' : V_2 \times V_1 \to W$ defined by $T'(v_2, v_1) = T(v_1, v_2)$. So it suffices to establish an isomorphism $\mathcal{BL}(V_1, V_2; W) \cong \mathcal{L}(V_1, \mathcal{L}(V_2, W))$.

Well, suppose that $T \in \mathcal{BL}(V_1, V_2; W)$. Then for any v_1, the function $T(v_1, -) : V_2 \to W$ is linear. In other words, $T^\sharp : v_1 \mapsto T(v_1, -)$ is a function $V_1 \to \mathcal{L}(V_2, W)$.

We first claim that $T^\sharp$ is linear, i.e. it is an element of $\mathcal{L}(V_1, \mathcal{L}(V_2, W))$. Unpacked, we want to show the following: given $v_1, v_1' \in V_1$ and $\lambda \in \mathbb{F}$, we want to show that we have an equality

$$T^\sharp(\lambda v_1 + v_1') \stackrel{?}{=} \lambda T^\sharp(v_1) + T^\sharp(v_1')$$

of elements of $\mathcal{L}(V_2, W)$. In other words, this should be an equality of functions, and two functions are equal if they have the same values. So we want to show that for every $v_2 \in V_2$,

$$T^\sharp(\lambda v_1 + v_1')(v_2) \stackrel{?}{=} \lambda T^\sharp(v_1)(v_2) + T^\sharp(v_1')(v_2)$$

is an equality in W. Except the LHS is $T(\lambda v_1 + v_1', v_2)$ and the right-hand side is $\lambda T(v_1, v_2) + T(v_1', v_2)$, and these are equal because the function $T(-, v_2)$ is linear.

So we have produced an operation $\sharp : \mathcal{BL}(V_1, V_2; W) \to \mathcal{L}(V_1, \mathcal{L}(V_2, W))$ sending $T \mapsto T^\sharp$. We next need to show that $\sharp$ is linear. In other words, we need to show that if $T, T' \in \mathcal{BL}(V_1, V_2; W)$ and $\lambda \in \mathbb{F}$, then

$$(\lambda T + T')^\sharp \stackrel{?}{=} \lambda T^\sharp + (T')^\sharp$$

is an equality in $\mathcal{L}(V_1, \mathcal{L}(V_2, W))$. But the left-hand side is the function $(\lambda T + T')(v_1, -) : V_2 \to W$ and the right-hand side is the function $\lambda T(v_1, -) + T'(v_1, -) : V_2 \to W$. In other words, we need to show that for any $v_2 \in V_2$, we have

$$(\lambda T + T')(v_1, v_2) \stackrel{?}{=} \lambda T(v_1, v_2) + T'(v_1, v_2)$$

but this is just how addition and scalar multiplication of functions is defined.

Summarizing so far, we have produced a linear transformation $\sharp : \mathcal{BL}(V_1, V_2; W) \to \mathcal{L}(V_1, \mathcal{L}(V_2, W))$. We now must show that it is an isomorphism. We will do this by exhibiting its inverse function.

Suppose that $S \in \mathcal{L}(V_1, \mathcal{L}(V_2, W))$. In other words, for any v_1, $S(v_1) \in \mathcal{L}(V_2, W)$ is a linear map. Define $S^\flat : V_1 \times V_2 \to W$ by $S^\flat(v_1, v_2) = S(v_1)(v_2)$. We claim that $S^\flat$ is bilinear. First, fixing v_1, we definitely have $S^\flat(v_1, -) = S(v_1) : V_2 \to W$ is a linear map, so that does half of the bilinearity. Next, fix v_2; then we claim that $S^\flat(-, v_2) : V_1 \to W$ is linear. Unpacked, we are claiming that, for any $v_1, v_1' \in V_1$ and any $\lambda \in \mathbb{F}$, then

$$S^\flat(\lambda v_1 + v_1', v_2) \stackrel{?}{=} \lambda S^\flat(v_1, v_2) + S^\flat(v_1', v_2)$$

is an equality in W. In other words, we are claiming that

$$S^\flat(\lambda v_1 + v_1', -) \stackrel{?}{=} \lambda S^\flat(v_1, -) + S^\flat(v_1', -)$$

is an equality of functions $V_2 \to W$. But the left-hand side is precisely $S(\lambda v_1 + v_1') \in \mathcal{L}(V_2, W)$ and the right-hand side is precisely $\lambda S(v_1) + S(v_1') \in \mathcal{L}(V_2, W)$, and these are equal by linearity of S.

Summarizing so far, we have a linear transformation $\mathcal{BL}(V_1, V_2; W) \to \mathcal{L}(V_1, \mathcal{L}(V_2, W))$ sending $T \mapsto T^\sharp$ defined by $T^\sharp(v_1)(v_2) = T(v_1, v_2)$, and we have a function $\mathcal{L}(V_1, \mathcal{L}(V_2, W)) \to \mathcal{BL}(V_1, V_2; W)$ sending $S \mapsto S^\flat$ defined by $S^\flat(v_1, v_2) = S(v_1)(v_2)$. Then we obviously have $(T^\sharp)^\flat = T$ and $(S^\flat)^\sharp = S$, and so these two transformations are inverses. Since one was linear, both are, and they give the desired isomorphism.

(d) **Suppose that V_1 and V_2 are finite dimensional. Assume that there exists a vector space V such that, for any vector space W, the vector spaces $\mathcal{BL}(V_1, V_2; W)$ and $\mathcal{L}(V, W)$ are isomorphic. What is the dimension of V?**

Remark: In fact, such a V does exist. It is called the *tensor product* of V_1 with V_2, and denoted $V_1 \otimes V_2$.

If there is supposed to be an isomorphism $\mathcal{BL}(V_1, V_2; W) \cong \mathcal{L}(V, W)$ for any W, then in particular there should be an isomorphism for $W = \mathbb{F}$. But $\dim \mathcal{L}(V, \mathbb{F}) = \dim V$, whereas, using (c), we learn that

$$\dim \mathcal{BL}(V_1, V_2; \mathbb{F}) = \dim \mathcal{L}(V_1, \mathcal{L}(V_2, \mathbb{F})) = \dim V_1 \times \dim \mathcal{L}(V_2, \mathbb{F}) = \dim V_1 \times \dim V_2.$$

So we must have $\boxed{\dim(V_1 \otimes V_2) = \dim V_1 \times \dim V_2}$.