Math 2135: Linear Algebra

Assignment 4

due 17 February 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. Let V and W be vector spaces, and let $T : V \to W$ be any function. The *graph* of T is defined to be the subset $\mathrm{graph}(T) \subset V \times W$ defined by

$$\mathrm{graph}(T) := \{(v, Tv) \text{ s.t. } v \in V\}.$$

   Prove that T is linear if and only if $\mathrm{graph}(T)$ is a vector subspace of $V \times W$.

2. Let V_1, V_2, W be vector spaces.
   (a) Find an isomorphism between $\mathcal{L}(W, V_1 \times V_2)$ and $\mathcal{L}(W, V_1) \times \mathcal{L}(W, V_2)$.
   (b) Find an isomorphism between $\mathcal{L}(V_1 \times V_2, W)$ and $\mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W)$.

3. Let V be a vector space over $\mathbb{F}$. Prove that a nonempty subset $X \subset V$ is an affine subspace (for some vector subspace $U \subset V$) if and only if, for any $v, w \in X$ and for any $\lambda \in \mathbb{F}$, $\lambda v + (1 - \lambda) w \in X$.

4. Let $U_1, U_2 \subset V$ be vector subspaces. Suppose that $X_1 \subset V$ is an affine subspace for U_1 and that $X_2 \subset V$ is an affine subspace for U_2. Prove that $X_1 \cap X_2$ is either empty or an affine subspace for $U_1 \cap U_2$.

5. Let $U \subset V$ be a vector subspace, let $\iota : U \to V$ denote the "identity" map $\iota(u) = u$, and let $\pi : V \to V/U$ the quotient map. Let W be another vector space, and consider the maps

$$\circ\iota : \mathcal{L}(V, W) \to \mathcal{L}(U, W),\ T \mapsto T \circ \iota,$$
$$\circ\pi : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W),\ S \mapsto S \circ \pi.$$

   Finally, let $X \subset \mathcal{L}(V, W)$ denote the subset

$$X := \{T \in \mathcal{L}(V, W) \text{ s.t. } \ker(T) \supseteq U\}.$$

   (a) Show that $\circ\iota$ and $\circ\pi$ are linear, and that X is a vector subspace.
   (b) Show that $\circ\pi$ is injective.
   (c) Show that $\mathrm{im}(\circ\pi) = X$. Conclude that $\mathcal{L}(V/U, W) \cong X \subset \mathcal{L}(V, W)$.
   (d) Show that $\ker(\circ\iota) = X$.

(e) Assuming that V is finite-dimensional, show that $\circ\iota$ is surjective. (In fact, $\circ\iota$ is surjective even if V is infinite-dimensional, but proving this requires some set-theoretic results that go beyond the class.) Conclude that $\mathcal{L}(U, W) \cong \mathcal{L}(V, W)/X$.

To summarize these results: $\mathcal{L}(-, W)$ takes subs to quotients and quotients to subs.

6. Let V_1, V_2, W be vector spaces. A function $T : V_1 \times V_2 \to W$ is called *bilinear* if for any $v_1 \in V_1$, the function $T(v_1, -) : V_2 \to W$ defined by $v_2 \mapsto f(v_1, v_2)$ is linear, and also for any $v_2 \in V_2$ the function $T(-, v_2) : V_1 \to W$ defined by $v_1 \mapsto T(v_1, v_2)$ is linear. Let $\mathcal{BL}(V_1, V_2; W)$ denote the set of bilinear functions $V_1 \times V_2 \to W$.

   (a) When $V_1 = V_2 = W = \mathbb{R}$, show that the addition function $A(v_1, v_2) := v_1 + v_2$ is linear but not bilinear, whereas the multiplication function $M(v_1, v_2) := v_1 v_2$ is bilinear but not linear.

   (b) Show that $\mathcal{BL}(V_1, V_2; W)$ is a vector subspace of the set $W^{V_1 \times V_2}$ of all functions.

   (c) Show that $\mathcal{BL}(V_1, V_2; W)$ is isomorphic to $\mathcal{L}(V_1, \mathcal{L}(V_2, W))$ and also to $\mathcal{L}(V_2, \mathcal{L}(V_1, W))$.

   (d) Suppose that V_1 and V_2 are finite dimensional. Assume that there exists a vector space V such that, for any vector space W, the vector spaces $\mathcal{BL}(V_1, V_2; W)$ and $\mathcal{L}(V, W)$ are isomorphic. What is the dimension of V?

   Remark: In fact, such a V does exist. It is called the *tensor product* of V_1 with V_2, and denoted $V_1 \otimes V_2$.