Math 2135: Linear Algebra

Assignment 5

Solutions

1. **Let V be a vector space and $T \in \mathcal{L}(V)$, and suppose that $T^n = 0$ for some positive integer n. Prove that $I - T$ is invertible and that its inverse is**

$$(I-T)^{-1} = I + T + T^2 + \cdots + T^{n-1}.$$

   We multiply

$$\begin{aligned}
&(I-T)(I+T+T^2+\cdots+T^{n-1}) \\
&\qquad = I - T + T - T^2 + T^2 - T^3 + \cdots + T^{n-2} - T^{n-1} + T^{n-1} - T^n \\
&\qquad\qquad = I + 0 + 0 + \cdots + 0 - T^n = I
\end{aligned}$$

   since $T^n = 0$. The same calculation shows that $(I+T+T^2+\cdots+T^{n-1})(I-T) = I$ (the two factors commute). So they are each other's inverses, and in particular both are invertible.

2. **Let V be a vector space and $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of T^2 if and only if at least one of 3 and -3 is an eigenvalue of T.**

   If 9 is an eigenvalue of T^2, then $T^2 - 9$ is not injective. But $T^2 - 9 = (T-3)(T+3)$, and the composition of injective operators is again injective. So at least one of $T-3$ and $T+3$ must not be injective, so at least one of 3 and -3 must be an eigenvalue of T.

   If 3 is an eigenvalue of T, let $v \neq 0$ be a corresponding eigenvector. Then $T^2v = TTv = T3v = 3Tv = 3^2v = 9v$, and so 9 is an eigenvalue of T^2. Similarly, if -3 is an eigenvalue of T, let $v \neq 0$ be a corresponding eigenvector, and compute $T^2v = (-3)^2v = 9v$.

3. **Let V be a vector space and $T \in \mathcal{L}(V)$, and suppose that $u, v \in V$ are eigenvectors of T such that $u + v$ is also an eigenvector. Prove that the eigenvalues of u and v are equal.**

   Suppose λ, μ are the eigenvalues of u, v respectively. If $\lambda \neq \mu$, then u and v are linearly independent. Compute

$$T(u+v) = Tu + Tv = \lambda u + \mu v.$$

   Since u, v are linearly independent, if $\lambda u + \mu v = \alpha u + \beta v$, then $\lambda = \alpha$ and $\mu = \beta$. In particular, since $\lambda \neq \mu$, we cannot have an equality between $\lambda u + \mu v$ and $\alpha(u+v) = \alpha u + \alpha v$ for any α.

   In other words, if u, v have different eigenvalues, then $u + v$ cannot be an eigenvector.

4. **Let V be a vector space and $S, T \in \mathcal{L}(V)$ two operators on V such that $ST = TS$. Show that $\ker(S)$ is invariant under T.**

   If $u \in \ker(S)$, then, using commutativity, we see that $STu = TSu = T0 = 0$, and so $Tu \in \ker(S)$.

5. **Let V be an n-dimensional vector space and $S, T \in \mathcal{L}(V)$ two operators on V such that $ST = TS$. Suppose that S has n distinct eigenvalues. Show that T is diagonalizable.**

   Hint: Show that any eigenbasis of S is also an eigenbasis of T.

   Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of S, and $v_1, \ldots, v_n$ some choice of corresponding eigenvectors (of S). We proved in class (and in the book) that $\{v_1, \ldots, v_n\}$ are linearly independent and hence a basis (since we are in an n-dimensional vector space). We claim that this basis is an eigenbasis for T. For this, it suffices to show that each v_i is an eigenvector for T.

   But, using commutativity, $STv_i = TSv_i = T\lambda_i v_i = \lambda_i T v_i$. In other words, $\ker(S - \lambda_i)$, the λ_ith eigenspace of S, is invariant under T. Since S has n distinct eigenvalues, its nontrivial eigenspaces are all one-dimensional. Thus $\ker(S - \lambda_i) = \mathrm{span}(v_i)$, and so $Tv_i = \mu_i v_i$ for some scalar μ_i. This is what we wanted to prove.

6. **Suppose that $c_1, \ldots, c_n \in \mathbb{R}$ are distinct real numbers. Prove that the functions $e^{c_1 x}, \ldots, e^{c_n x}$ are linearly independent in the vector space $\mathbb{R}^{\mathbb{R}}$.**

   Hint: Let $V := \mathrm{span}(e^{c_1 x}, \ldots, e^{c_n x})$ and define an operator $T \in \mathcal{L}(V)$ by $T[f] = f'$, or in other words $T = \frac{\mathrm{d}}{\mathrm{d}x}$. What are its eigenvalues and eigenvectors?

   Since sums and products of smooth functions are smooth, and since exponential functions are smooth, every function in V is smooth. Moreover, $T[e^{c_i x}] = c_i e^{c_i x} \in V$. So T is a well-defined linear operator $V \to V$. But $e^{c_i x}$ is an eigenvector under T with eigenvalue c_i. Since the c_i's are distinct, the $e^{c_i x}$'s must be linearly independent.

   Remark: One can also show the linear independence as follows. Order the numbers $c_1, \ldots, c_n$ in increasing order $c_1 < c_2 < \cdots < c_n$. Now suppose that we can find $\alpha_1, \ldots, \alpha_n$ such that $0 = \alpha_1 e^{c_1 x} + \cdots + \alpha_n e^{c_n x}$ for all x. Suppose that $\alpha_n \neq 0$. Let $x > \max_{i=1,\ldots,n-1}\Big(\frac{1}{c_n - c_i} \log\big(\frac{(n-1)|\alpha_i|}{|\alpha_n|}\big)\Big)$. Then $|\alpha_n e^{c_n x}| > |\alpha_1 e^{c_1 x}| + \cdots + |\alpha_n e^{c_n x}| > |\alpha_1 e^{c_1 x} + \cdots + \alpha_n e^{c_n x}|$, violating the functional equation $0 = \alpha_1 e^{c_1 x} + \cdots + \alpha_n e^{c_n x}$. So $\alpha_n = 0$. Now induct: use the same argument to show that $\alpha_{n-1} = 0$, and so on.

   The advantage of the first proof is that it works with $\mathbb{R}$ replaced by $\mathbb{C}$.