Math 2135: Linear Algebra

Assignment 5

due 18 March 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. Let V be a vector space and $T \in \mathcal{L}(V)$, and suppose that $T^n = 0$ for some positive integer n. Prove that $I - T$ is invertible and that its inverse is

$$(I - T)^{-1} = I + T + T^2 + \cdots + T^{n-1}.$$

2. Let V be a vector space and $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of T^2 if and only if at least one of 3 and -3 is an eigenvalue of T.

3. Let V be a vector space and $T \in \mathcal{L}(V)$, and suppose that $u, v \in V$ are eigenvectors of T such that $u + v$ is also an eigenvector. Prove that the eigenvalues of u and v are equal.

4. Let V be a vector space and $S, T \in \mathcal{L}(V)$ two operators on V such that $ST = TS$. Show that $\ker(S)$ is invariant under T.

5. Let V be an n-dimensional vector space and $S, T \in \mathcal{L}(V)$ two operators on V such that $ST = TS$. Suppose that S has n distinct eigenvalues. Show that T is diagonalizable.

   Hint: Show that any eigenbasis of S is also an eigenbasis of T.

6. Suppose that $c_1, \ldots, c_n \in \mathbb{R}$ are distinct real numbers. Prove that the functions $e^{c_1 x}, \ldots, e^{c_n x}$ are linearly independent in the vector space $\mathbb{R}^{\mathbb{R}}$.

   Hint: Let $V := \operatorname{span}(e^{c_1 x}, \ldots, e^{c_n x})$ and define an operator $T \in \mathcal{L}(V)$ by $T[f] = f'$, or in other words $T = \frac{\mathrm{d}}{\mathrm{d}x}$. What are its eigenvalues and eigenvectors?