Math 2135: Linear Algebra

Assignment 6

Solutions

Primary questions

1. **Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 + 1 = 0$.**

   For example, rotation by $45°$.

2. **Is the matrix**
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$
   diagonalizable? Justify your answer.

   This operator is not diagonalizable. To see this, note that its only eigenvalues are 1 and 5, and the corresponding eigenspaces are both 1-dimensional.

3. (a) **Suppose that $R, T \in \mathcal{L}(\mathbb{F}^3)$ each have $2, 6, 7$ as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{F}^3)$ such that $R = STS^{-1}$.**

   We can find eigenbases for both of these matrices, since they have all-distinct eigenvalues. So both are conjugate to the diagonal matrix with diagonal entries $2, 6, 7$, and hence they are conjugate to each other.

   (b) **Find a pair of operators $R, T \in \mathcal{L}(\mathbb{F}^4)$ with the following properties: each of them have $2, 6, 7$ as eigenvalues and neither has any other eigenvalues; there does not exist an an invertible operator $S \in \mathcal{L}(\mathbb{F}^4)$ such that $R = STS^{-1}$.**

   For example, the diagonal matrices
$$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 6 & \\ & & & 7 \end{pmatrix}, \begin{pmatrix} 2 & & & \\ & 6 & & \\ & & 6 & \\ & & & 7 \end{pmatrix}.$$
   They cannot be conjugate, since their eigenspaces are of different dimensions.

4. **Suppose that $T \in \mathcal{L}(\mathbb{F}^5)$ and that the eigenspace $E(8, T)$ is 4-dimensional. Show that at least one of $T - 2I$ and $T - 6I$ is invertible.**

   The sum of eigenspaces is always direct. Thus the eigenspaces $E(2, T)$ and $E(6, T)$ cannot both be at-least-one-dimensional since $4 + 1 + 1 > 5$.

5. **Suppose $T \in \mathcal{L}(V)$. Prove that $T/\operatorname{im}(T) = 0$.**

   By definition, $T/\operatorname{im}(T)$ is the operator in $V/\operatorname{im}(T) = \{v + \operatorname{im}(T) : v \in V\}$ that takes $v + \operatorname{im}(T)$ to $Tv + \operatorname{im}(T)$. But $Tv + \operatorname{im}(T) = \operatorname{im}(T)$, since $Tv \in \operatorname{im}(T)$, and $\operatorname{im}(T) = 0 + \operatorname{im}(T) \in V/\operatorname{im}(T)$

is the zero vector. So $T/\operatorname{im}(T)$ is the operator on $V/\operatorname{im}(T)$ that takes everything to 0, and so it is the zero operator.

6. **Suppose that V is a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $U \subset V$ is a T-invariant vector subspace. Show that each eigenvalue of T/U is also an eigenvalue of T.**

   Let λ be an eigenvalue of T/U. Note that $(T/U) - \lambda = (T - \lambda)/U$; λ being an eigenvalue of T/U is the statement that this operator is not injective. Since V/U is finite-dimensional (its dimension is at most $\dim V$), non-injectivity of an operator on V/U is equivalent to non-surjectivity. So let $v + U \in V/U$ be an element which is not in the image of $(T - \lambda)/U$. In other words, there does not exist a $w \in V$ such that $(T - \lambda)w + U = v + U$. Then certainly there is no w solving $(T - \lambda)w = v$. So v is not in the image of $T - \lambda$, and so $T - \lambda$ is not surjective. Since V is finite-dimensional, non-surjectivity of an operator on V is equivalent to non-injectivity. So $T - \lambda$ is not injective, and λ is an eigenvalue of T.

Bonus questions

7. **Show that the finite-dimensionality is necessary in exercise 6. Specifically, find a (necessarily infinite-dimensional) vector space V and an operator $T \in \mathcal{L}(V)$ such that T has no eigenvalues at all, but such that there is an invariant subspace U for which T/U has an eigenvalue.**

   For example, let V be the vector space $\mathbb{F}[x]$ of polynomials, and $T = \hat{x}$ the operator which multiplies a polynomial by x. Then T cannot have any eigenvalues, because if $f(x) \in \mathbb{F}[x]$ is nonzero, then the degree of Tf is one more than the degree of λf (and so they cannot be equal).

   Let $U = \operatorname{im}(T)$. Then $1 \notin U$, and so V/U is not the zero vector space. By exercise 5, $T/U = 0$, which has 0 as an eigenvalue.

8. **In exercises 6 and 7, what would happen if you used generalized eigenvalues rather than eigenvalues?**

   The statement still fails. Let $V = \mathbb{R}^{\mathbb{R}}$ be the space of all functions, and T the operator that shifts $f(x) \mapsto f(x-1)$. Then T is invertible, and so 0 is not a generalized eigenvalue of T.

   Let U be the subspace of functions which vanish on negative values of x. In other words

   $$U = \{f \in \mathbb{R}^{\mathbb{R}} \text{ s.t. } f(x) = 0 \forall x < 0\}.$$

   Then we claim that U is T-invariant. Indeed, suppose $f \in U$ and $x < 0$. Then $x - 1 < 0$ so $f(x-1) = 0$. So the function Tf, defined by $(Tf)(x) = f(x-1)$, is in U.

   Note that $T|_U$ is not invertible. Indeed, its image consists of the functions which vanish on $x < 1$.

   What is the quotient space V/U? We can choose a splitting $V = U \oplus W$ where $W = \{f \in \mathbb{R}^{\mathbb{R}} \text{ s.t. } f(x) = 0 \forall x \geq 0\}$. This splitting selects an isomorphism $V/U \cong W$. Under this isomorphism, the quotient operator T/U sends a function f to the function Tf defined by

   $$(Tf)(x) = \begin{cases} f(x-1), & x < 0 \\ 0, & x \geq 0 \end{cases}$$

This operator is not invertible: its kernel consists of the functions supported in the interval $(0, 1]$.

Thus 0 is a generalized eigenvalue of T/U (indeed, an actual eigenvalue). But 0 was not an eigenvalue of T.