Math 2135: Linear Algebra

Assignment 6

due 25 March 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

Primary questions

1. Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 + 1 = 0$.

2. Is the matrix
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$
diagonalizable? Justify your answer.

3. (a) Suppose that $R, T \in \mathcal{L}(\mathbb{F}^3)$ each have $2, 6, 7$ as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{F}^3)$ such that $R = STS^{-1}$.

   (b) Find a pair of operators $R, T \in \mathcal{L}(\mathbb{F}^4)$ with the following properties: each of them have $2, 6, 7$ as eigenvalues and neither has any other eigenvalues; there does not exist an an invertible operator $S \in \mathcal{L}(\mathbb{F}^4)$ such that $R = STS^{-1}$.

4. Suppose that $T \in \mathcal{L}(\mathbb{F}^5)$ and that the eigenspace $E(8, T)$ is 4-dimensional. Show that at least one of $T - 2I$ and $T - 6I$ is invertible.

5. Suppose $T \in \mathcal{L}(V)$. Prove that $T/\operatorname{im}(T) = 0$.

6. Suppose that V is a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $U \subset V$ is a T-invariant vector subspace. Show that each eigenvalue of T/U is also an eigenvalue of T.

Bonus questions

7. Show that the finite-dimensionality is necessary in exercise 6. Specifically, find a (necessarily infinite-dimensional) vector space V and an operator $T \in \mathcal{L}(V)$ such that T has no eigenvalues at all, but such that there is an invariant subspace U for which T/U has an eigenvalue.

8. In exercises 6 and 7, what would happen if you used generalized eigenvalues rather than eigenvalues?