Math 2135: Linear Algebra

Assignment 7

due 5 April 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. (a) **Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (z, 0)$. Find the generalized eigenspaces of T.**
   We first note that $T^2 = 0$. This implies that the only possible eigenvalue is $\lambda = 0$. It also means that the entire $\mathbb{C}^2$ is the generalized eigenspace $G(0, T)$.

   (b) **Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (-z, w)$. Find the generalized eigenspaces of T.**
   We first note that $T^2 = -1$. This implies that the possible eigenvalues are $\pm\sqrt{-1}$. We note that $(w, z) = (1, +\sqrt{-1})$ is an eigenvector with eigenvalue $-\sqrt{-1}$, whereas $(1, -\sqrt{-1})$ is an eigenvector with eigenvalue $+\sqrt{-1}$. Thus both $\pm\sqrt{-1}$ are eigenvalues. Then the generalized eigenspaces cannot be more than 1-dimensional, and we find that $G(+\sqrt{-1}, T) = \operatorname{span}((1, -\sqrt{-1}))$ and $G(-\sqrt{-1}, T) = \operatorname{span}((1, +\sqrt{-1}))$.

2. **Suppose that $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that**
$$v, Tv, T^2v, \dots, T^{m-1}v$$
is linearly independent.

   Consider a solution $\alpha_0, \dots, \alpha_{m-1} \in \mathbb{C}$ to the equation
$$(*) \quad 0 = \alpha_0 v + \alpha_1 Tv + \cdots + \alpha_{m-1}T^{m-1}v.$$
   Acting on both sides by T^{m-1} would give
$$0 = \alpha_0 T^{m-1}v + 0$$
   since $T^m v = 0$. But since $T^{m-1}v \neq 0$, we find that $\alpha_0 = 0$.

   In this case, we can act on both sides of equation $(*)$ by T^{m-2} to produce
$$0 = 0T^{m-2}v + \alpha_1 T^{m-1}v + 0.$$
   But this implies that $\alpha_1 = 0$.

   In this case, we can act on both sides of equation $(*)$ by T^{m-3} and produce
$$0 = 0T^{m-3}v + 0T^{m-2}v + \alpha_2 T^{m-1}v + 0.$$
   But this implies that $\alpha_2 = 0$.

   Continuing in this way, we find that all the α's must be zero, and so there cannot be a linear dependence.

3. **Let $T \in \mathcal{L}(\mathbb{C}^3)$ be the operator defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T does not have a square root. In other words, prove that there is no operator $S \in \mathcal{L}(\mathbb{C}^3)$ such that $S^2 = T$.**

   Suppose for contradiction that there were such an S. If λ is an eigenvalue of S, then λ^2 will be an eigenvalue of $S^2 = T$. But the only eigenvalue of T is 0. Since the only square root of 0 is 0, we find that S must have 0 as its only eigenvalue.

   This implies that $S^3 = 0$, since $\mathbb{C}^3$ is 3-dimensional. But then $T^2 = SS^3 = 0$. However, $T^2 \neq 0$, providing the desired contradiction.

4. **Suppose that $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are 3, 5, and 8. Prove that $(T-3)^2(T-5)^2(T-8)^2 = 0$.**

   The generalized eigenspaces $G(3,T)$, $G(5,T)$, and $G(8,T)$ are all at least one-dimensional, and their dimensions sum to 4. So precisely one of them is 2-dimensional. So the characteristic polynomial is one of: $(x-3)^2(x-5)(x-8)$, $(x-3)(x-5)^2(x-8)$, or $(x-3)(x-5)(x-8)^2$. Thus at least one of the following three equations holds:

$$
\begin{aligned}
(T-3)^2(T-5)(T-8) &= 0 \\
(T-3)(T-5)^2(T-8) &= 0 \\
(T-3)(T-5)(T-8)^2 &= 0
\end{aligned}
$$

   The desired equation follows from any of these three choices.

5. **Suppose that V is a finite-dimensional vector space over $\mathbb{F} = \mathbb{R}$. Show that there exists an operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$ if and only if $\dim V$ is even.**

   Note that if $T^2 = -I$, then T has no real eigenvalues. We proved that this is impossible in odd dimensions (since a real polynomial of odd degree has a real solution).

   If $\dim V$ is even, then for example we can (choose a basis and) take the operator (which when written in that basis looks like):

$$
T = \begin{pmatrix}
0 & 1 & & & & & \\
-1 & 0 & & & & & \\
& & 0 & 1 & & & \\
& & -1 & 0 & & & \\
& & & & \ddots & & \\
& & & & & 0 & 1 \\
& & & & & -1 & 0
\end{pmatrix}
$$

6. **Hint for this question: Read Theorem 8.31 on page 258 of *Linear Algebra Done Right*.**

   Let V be a finite-dimensional vector space and suppose that $T \in \mathcal{L}(V)$ is nilpotent. Consider the following formulas:

$$
\exp(T) := \sum_{k=0}^{\infty} \frac{1}{k!} T^k, \qquad \log(1+T) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} T^k.
$$

   (a) **Explain why, even though the formulas look like infinite sums, actually these sums are finite.**

   Since T is nilpotent, there is some number n such that $T^n = 0$. Then every term in these sums with $k \geq n$ is zero, and so the sums truncate.

(b) **Show that** $\log(1+T)$ **is nilpotent. Show that** $\exp(T)-1$ **is nilpotent. Hint: Show that if** T **is nilpotent and** $ST=TS$**, then** ST **is nilpotent.**

Both $\log(1+T)$ and $\exp(T)-1$ are of the form $Tp(T)$ for some polynomials $p(x)$. But $(Tp(T))^n = T^n p(T)^n = 0$.

(c) **Show** $\exp(\log(1+T)) = 1+T$ **and that** $\log(\exp(T)) = T$**. Hint: Use the fact that the Taylor series of** e^x **is** $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ **and that the Taylor series of** $\ln(1+x)$ **is** $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$**.**

Take the equations $e^{\ln(1+x)} = 1+x$ and $\ln(e^x) = \ln(1+(e^x-1)) = x$, and Taylor-expand them at $x=0$. You will get equations of formal power series. Now plug in T and note that $T^k = 0$ for all $k \geq n$.