Math 2135: Linear Algebra

Assignment 7

due 5 April 2022, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You should attempt every question, but it is not expected that you will solve all of them.

1. (a) Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w,z) = (z,0)$. Find the generalized eigenspaces of T.
   (b) Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w,z) = (-z,w)$. Find the generalized eigenspaces of T.

2. Suppose that $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that
$$v, Tv, T^2v, \ldots, T^{m-1}v$$
is linearly independent.

3. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be the operator defined by $T(z_1,z_2,z_3) = (z_2,z_3,0)$. Prove that T does not have a square root. In other words, prove that there is no operator $S \in \mathcal{L}(\mathbb{C}^3)$ such that $S^2 = T$.

4. Suppose that $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are 3, 5, and 8. Prove that $(T-3)^2(T-5)^2(T-8)^2 = 0$.

5. Suppose that V is a finite-dimensional vector space over $\mathbb{F} = \mathbb{R}$. Show that there exists an operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$ if and only if $\dim V$ is even.

6. **Hint for this question:** Read Theorem 8.31 on page 258 of *Linear Algebra Done Right*.

   Let V be a finite-dimensional vector space and suppose that $T \in \mathcal{L}(V)$ is nilpotent. Consider the following formulas:
$$\exp(T) := \sum_{k=0}^{\infty} \frac{1}{k!}T^k, \qquad \log(1+T) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}T^k.$$
   (a) Explain why, even though the formulas look like infinite sums, actually these sums are finite.
   (b) Show that $\log(1+T)$ is nilpotent. Show that $\exp(T) - 1$ is nilpotent. **Hint:** Show that if T is nilpotent and $ST = TS$, then ST is nilpotent.
   (c) Show $\exp(\log(1+T)) = 1+T$ and that $\log(\exp(T)) = T$. **Hint:** Use the fact that the Taylor series of e^x is $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ and that the Taylor series of $\ln(1+x)$ is $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}x^k$.