Math 2135: Linear Algebra

Midterm exam

Solutions

Part A.

The next three pages presents you with six statements. Every one of them is false. For each one, give a short (one or two sentences) explanation of why.

1. **The empty set is a vector space.**

   Every space is required to contain a zero vector.

2. **If V is a vector space and $\{v_1, \dots, v_m\}$ is a linearly independent set of vectors in V, and $\{w_1, \dots, w_n\}$ is a spanning set of V, then $m < n$.**

   It is always true that $m \leq n$, but the strict inequality might not hold.

3. **There exists a linear transformation $\mathbb{R}^7 \to \mathbb{R}^4$ with two-dimensional kernel.**

   If $T : \mathbb{R}^7 \to \mathbb{R}^4$ is a linear transformation, then $\dim(\operatorname{im} T) \leq 4$, and so $\dim(\ker T) = \dim(\mathbb{R}^7) - \dim(\operatorname{im} T) = 7 - \dim(\operatorname{im} T) \geq 3$.

4. **Suppose $T : V \to W$ is a linear transformation, and $\{v_1, \dots, v_n\}$ is a basis for V. If $\{Tv_1, \dots, Tv_n\}$ spans W, then T is injective.**

   If $\{Tv_1, \dots, Tv_n\}$ spans W, then T is surjective, but not necessarily injective. Injectivity of T is equivalent to $\{Tv_1, \dots, Tv_n\}$ being linearly independent.

5. **Suppose V_1 and V_2 are finite-dimensional vector space. Then $\dim(V_1 \times V_2) = \dim(V_1) \times \dim(V_2)$.**

   The correct formula is $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$.

6. **If $T : V \to W$ is a linear transformation, and $U \subseteq V$ is a vector subspace, then there exists a linear transformation $\bar{T} : V/U \to W$ defined by $\bar{T}(v + U) = T(v)$.**

   This formula for $\bar{T}$ only defines a function $V \to W$ if $\ker T \supseteq U$.

Part B.

Let $V = \mathbb{R}^{\leq 2}[x]$ be the vector space of polynomials of degree ≤ 2. (This vector space is denoted $\mathcal{P}_2(\mathbb{R})$ in the textbook.) For each real number $r \in \mathbb{R}$, let $\mathrm{ev}_r : V \to \mathbb{R}$ denote the linear functional that sends $f(x) \mapsto f(r)$.

Show that the set $\{x^2-1, x^2-x, x^2+x\}$ is a basis for V, and that the set $\{-\mathrm{ev}_0, \frac{1}{2}\mathrm{ev}_{-1}, \frac{1}{2}\mathrm{ev}_1\}$ is its dual basis.

We know that $\dim V = 3$ (e.g. because the set $\{1, x, x^2\}$ is a basis). So to show that $\{x^2 - 1, x^2 - x, x^2 + x\}$ is a basis for V, it suffices to show that it is linearly independent. Suppose that $a, b, c \in \mathbb{R}$ such that

$$0 = a(x^2 - 1) + b(x^2 - x) + c(x^2 + x).$$

Unpacked, we see that

$$0 = -a + (c - b)x + (a + b + c)x^2.$$

But then $0 = -a = c - b = a + b + c$. Hence $a = 0$, and so $0 = c + b$. Together with $0 = c - b$, we see that $b = c = 0$.

Now we check the statement about dual bases. We need to check the following nine equations:

$$\begin{array}{lll}
-\mathrm{ev}_0(x^2-1) = 1 & \frac{1}{2}\mathrm{ev}_{-1}(x^2-1) = 0 & \frac{1}{2}\mathrm{ev}_1(x^2-1) = 0 \\
-\mathrm{ev}_0(x^2-x) = 0 & \frac{1}{2}\mathrm{ev}_{-1}(x^2-x) = 1 & \frac{1}{2}\mathrm{ev}_1(x^2-x) = 0 \\
-\mathrm{ev}_0(x^2+x) = 0 & \frac{1}{2}\mathrm{ev}_{-1}(x^2+x) = 0 & \frac{1}{2}\mathrm{ev}_1(x^2+x) = 1
\end{array}$$

In other words, we need to check that

$$\begin{array}{lll}
-(0^2-1) = 1 & \frac{1}{2}((-1)^2-1) = 0 & \frac{1}{2}(1^2-1) = 0 \\
-(0^2-0) = 0 & \frac{1}{2}((-1)^2-(-1)) = 1 & \frac{1}{2}(1^2-1) = 0 \\
-(0^2+0) = 0 & \frac{1}{2}((-1)^2+(-1)) = 0 & \frac{1}{2}(1^2+1) = 1
\end{array}$$

These are all in fact true.