Suppose given a field extension $\mathbb{F} \hookrightarrow \mathbb{E}$ $\rightsquigarrow$ $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) := \mathrm{Aut}_{\mathbb{F}}(\mathbb{E})$.

Then consider antimaps aka contravariant maps of posets.

$$\left\{\begin{array}{c}\text{sub extensions}\\ \text{of } \mathbb{E}\end{array}\right\} \rightleftarrows \left\{\text{sub gps of } \mathrm{Gal}(\mathbb{E}/\mathbb{F})\right\}$$

$$\begin{tikzcd}
\mathbb{F} & & & \mathrm{Gal}(\mathbb{E}/\mathbb{F}) \\
\cap & & \mathrm{Gal}(\mathbb{E}/\mathbb{K}) & \cup \\
\mathbb{K} \arrow[urr, mapsto] & \mathbb{E}^G & & G \arrow[ll, mapsto] \\
\cap & & & \cup \\
\mathbb{E} & & & 1
\end{tikzcd}$$

If $\mathbb{K} \subseteq \mathbb{L}$ then $\mathrm{Gal}(\mathbb{E}/\mathbb{L}) \subseteq \mathrm{Gal}(\mathbb{E}/\mathbb{K})$.

If $H \subseteq G$ then $\mathbb{E}^G \subseteq \mathbb{E}^H$.

Start with

$$\begin{array}{c} \mathbb{F} \\ \cap \\ \mathbb{K} \\ \cap \\ \mathbb{E} \end{array} \quad \longmapsto \quad \mathrm{Gal}(\mathbb{E}/\mathbb{K}) \quad \longmapsto \quad \boxed{\begin{array}{c} \mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{K})} \\ \cup\!| \\ \mathbb{K} \end{array}}$$

$\leftarrow$ subfield of $\mathbb{E}$.

$\leftarrow$ $\mathrm{id}|_{\{\text{subextensions}\}}$

$\cap\!|$

composition

subexts $\to$ subgps $\to$ subexts

Posets are special types of categories.

map of posets $\equiv$ functor.

If X, Y both posets, then

$$\mathrm{maps}(X, Y) := \{ f : X \to Y \text{ s.t. if } x_1 \le x_2 \text{ then } f(x_1) \le f(x_2) \}$$

is again a poset: $f_1 \le f_2$ if $f_1(x) \le f_2(x)$ $\forall x$.

$$\begin{array}{ccccc} G & \longmapsto & \mathbb{E}^G & \longmapsto & \mathrm{Gal}(\mathbb{E}/\mathbb{E}^G) \\ \cap\!| & & & & \cup\!| \\ \mathrm{Gal}(\mathbb{E}/\mathbb{F}) & & & & G \end{array}$$

Again: composition $\supseteq$ id.

<u>Defn</u>: Given two posets X, Y, a <u>Galois connection</u> between them is a pair of contravariant maps

$$f : X \rightleftarrows Y : g$$

s.t. $\mathrm{id}_Y \leq fg$ and $\mathrm{id}_X \leq gf$.

If given contravariant $f : X \rightleftarrows Y : g$, $fg \geq id_Y$ and $gf \geq id_X$;

then $fgf = f$ and $gfg = g$

$$fg \circ f \geq id \circ f = f$$

$$f \circ \underbrace{gf}_{\geq id} \leq f \circ id = f$$

In our example: $\mathbb{K} \mapsto Gal(\mathbb{E}/\mathbb{K}) \mapsto \mathbb{E}^{Gal(\mathbb{E}/\mathbb{K})} \mapsto$

$$Gal(\mathbb{E}/\mathbb{K}) = Gal\left(\mathbb{E}/\mathbb{E}^{Gal(\mathbb{E}/\mathbb{K})}\right)$$

Define: $\mathbb{K} \subseteq \mathbb{E}$ is Galois if $\mathbb{K} = \mathbb{E}^{\mathrm{Gal}(\mathbb{E}/\mathbb{K})}$

$$\begin{array}{ccccccc} G & \longmapsto & \mathbb{E}^G & \longmapsto & \mathrm{Gal}(\mathbb{E}/\mathbb{E}^G) & \longmapsto & \mathbb{E}^{\mathrm{Gal}\ldots} \\ \cap\!\! | & & & & & & \\ \mathrm{Gal}(\mathbb{E}/\mathbb{F}) & & & & & & \end{array}$$

Prop: For any subgp $G \le \mathrm{Gal}(\mathbb{E}/\mathbb{F})$,

$\mathbb{E}^G \subseteq \mathbb{E}$ is a Galois.

Remark about categories:

Posets $\subseteq$ categories.

- set of "objects"
- notion of "$\leq$"
- transitivity.

yes or no question.

- set of objects
- given x, y objects, there is a set of ways for "$x \leq y$".
- transitive: ie given a way that $x \leq y$ and a way that $y \leq z$ get a way that $x \leq y$.

"composition"

If I'm given ways $x \leq y \leq z \leq w$, transitivity gives me two different ways $x \leq w$.

Requirement in a category: these a priori different proofs give the same way $x \leq w$.

associativity

If Given categories X, Y, a ~~dual~~ adjunction is a pair of covariant ~~contravariant~~ functors

f = left adj
g = right adj.
"$f \dashv g$"

$$f : X \rightleftarrows Y : g$$

and ways such that

$$id_Y \cancel{\leq} \geq fg \qquad id_X \leq gf$$

part of data of dual adjunction.

s.t. the induced way for $f \leq f$ for $g \leq g$ are the canonical "identity" ways.

"$g \vdash f$"

<u>Remark:</u> Suppose $f : X \to Y$ (of posets) is an contravariant equivalence.

i.e. $\exists$ $g : Y \to X$ another (contravariant) functor

and $\underbrace{fg = id_Y}_{\text{is in particular an inequality.}}$ $gf = id_X$.

$\rightsquigarrow$ $\underbrace{\text{contravariant equivalences}}$ are examples of $\underbrace{\text{dual adjunctions}}$

in poset case.

$\equiv$ Galois correspondence. $\equiv$ Galois connection

Fundamental Thm of Galois Theory:

Suppose $\mathbb{F} \subset \mathbb{E}$ is finite and Galois.

1. Then the Galois connection $\{\text{subexts of } \mathbb{E}\} \rightleftarrows \{\text{subgps of Gal}\}$ is an equivalence. In particular, every subext $\mathbb{K} \subseteq \mathbb{E}$ is Galois.

2. $\mathbb{F} \subset \mathbb{K}$ is Galois iff $\mathrm{Gal}(\mathbb{E}/\mathbb{K}) \subset \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ is $\boxed{\text{normal.}}$ in which case, $\mathrm{Gal}(\mathbb{K}/\mathbb{F}) =$ quotient gp $\mathrm{Gal}(\mathbb{E}/\mathbb{F}) / \mathrm{Gal}(\mathbb{E}/\mathbb{K})$.

3. $[\mathbb{E}:\mathbb{K}] = \# \mathrm{Gal}(\mathbb{E}/\mathbb{K})$ and $[\mathbb{K}:\mathbb{F}] = \#$ coset space $\mathrm{Gal}(\mathbb{E}/\mathbb{F})/\mathrm{Gal}(\mathbb{E}/\mathbb{K})$.