Math 5055

- HW 1 posted at categorified.net/22Winter5055/content.html
  Due next week.
- OH this week TTh 1-2:30
- "return" to in person Jan 31. LSC - Common Area C212.
- Start thinking about your presentation/paper topics.
- if you have notes you are willing to share, I'll post them on the website.

Today: Prove the Fundamental Thm of Galois Thy.

Fundamental Thm of Galois Thy

Let $\mathbb{F} \subset \mathbb{E}$ be finite Galois extension w/ Galois gp $G := \mathrm{Gal}(\mathbb{E}/\mathbb{F})$.

we'll discuss how to drop this.

Then

1. The Galois connection

$$\left\{\begin{array}{c}\text{intermediate subfields}\\ \mathbb{F} \subset \mathbb{K} \subset \mathbb{E}\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{subgps of}\\ G\end{array}\right\}$$

is a (anti) isomorphism of posets. i.e. it is a Galois correspondence.

2. If $\mathbb{F} \subset \mathbb{K} \subset \mathbb{E}$, $[\mathbb{E}:\mathbb{K}] = \# \mathrm{Gal}(\mathbb{E}/\mathbb{K})$.

3. If $\mathbb{F} \subset \mathbb{K} \subset \mathbb{E}$, then ① ⇒ $\mathbb{K} \subset \mathbb{E}$ is Galois, $\mathbb{F} \subset \mathbb{K}$ is Galois iff $\mathrm{Gal}(\mathbb{E}/\mathbb{K}) \subset G$ is normal, in which case $\mathrm{Gal}(\mathbb{K}/\mathbb{F}) = G/\mathrm{Gal}(\mathbb{E}/\mathbb{K})$.

Remark:

Given any Galois connection $f: X \rightleftarrows Y : g$ of posets

i.e. anti maps s.t.
$x \le gf(x)$
$y \le fg(y)$
$\forall x \in X, y \in Y$.

$x \in X$ is closed if $x = gf(x)$

e.g. $\mathbb{F} \subset \mathbb{K} \subset \mathbb{E}$, $\mathbb{K}$ is closed iff $\mathbb{K} \subset \mathbb{E}$ is Galois.

In the case of infinite algebraic Galois extensions $\mathbb{F} \subset \mathbb{E}$:

- every intermediate $\mathbb{K}$ is indeed closed
- $Gal(\mathbb{E}/\mathbb{F})$ has a natural topology, and the closed subgps are the closed subgps.

Lemma: Suppose $\mathbb{F} \subset \mathbb{E}$ any field extension, and

$$\begin{array}{ccc} \mathbb{F} & & \mathrm{Gal}(\mathbb{E}/\mathbb{F}) = G \\ \cap & & \cup \\ \mathbb{K} & \rightsquigarrow & \mathrm{Gal}(\mathbb{E}/\mathbb{K}) \\ \cap & & \cup \\ \mathbb{L} & & \mathrm{Gal}(\mathbb{E}/\mathbb{L}) \\ \cap & & \cup \\ \mathbb{E} & & \mathrm{Gal}(\mathbb{E}/\mathbb{E}) = \{1\} \end{array}$$

in $\mathbb{N} \cup \{\infty\}$.

Then $\underbrace{[\mathrm{Gal}(\mathbb{E}/\mathbb{K}) : \mathrm{Gal}(\mathbb{E}/\mathbb{L})]}_{\# \frac{\mathrm{Gal}(\mathbb{E}/\mathbb{K})}{\mathrm{Gal}(\mathbb{E}/\mathbb{L})}} \leq [\mathbb{L} : \mathbb{K}]$.

Pf: Triv if $[\mathbb{L}:\mathbb{K}] = \infty$. Triv if $[\mathbb{L}:\mathbb{K}] = 1$.

Pf (cont'd): So we will induct on

$1 < n := [\mathbb{L} : \mathbb{K}] < \infty$

Pick $u \in \mathbb{L} \setminus \mathbb{K}$. Let $p(x) \in \mathbb{K}[x]$

its minimal poly. $k := \deg(u)$.

Then $$\overbrace{\mathbb{K} \underset{k>1}{\subset} \mathbb{K}[u] \underset{n/k<n}{\subset} \mathbb{L}}^{n}.$$

If $\frac{n}{k} > 1$, then $k < n$, and we're done by

induction. Because if given $H < I \subset J$ inclusions

of gps, then $[J:H] = \underbrace{[J:I]}_{\leq \frac{n}{k}} \cdot \underbrace{[I:H]}_{\leq k}$.

So only case left to consider $\mathbb{L} = \mathbb{K}[u]$.

Care about $\# \dfrac{Gal(\mathbb{E}/\mathbb{K})}{Gal(\mathbb{E}/\mathbb{L})} \overset{?}{\leq} n = k.$

Given $\tau \in Gal(\mathbb{E}/\mathbb{K})$ representing coset $[\tau]$,

look at $\tau \circ u \in \mathbb{E}$. Again solves $p(x)$.

But if $[\tau] = [\tau']$ then $\tau = \tau' \circ \sigma$, $\sigma \in Gal(\mathbb{E}/\mathbb{L})$

$\tau \cdot u = \tau' \circ \underbrace{\sigma \cdot u} = \tau' \circ u$. i.e. we just built

a map $\dfrac{Gal(\mathbb{E}/\mathbb{K})}{Gal(\mathbb{E}/\mathbb{L})} \longrightarrow \underbrace{\text{roots of } p \text{ in } \mathbb{E}}_{\text{set of size } \leq n}.$

So it suffices to show that $[\tau] \mapsto \tau \circ u$ is an injection.

But if $\tau \circ u = \tau' \circ u$, then $\tau^{-1}(\tau') \circ u = u$

So $\tau^{-1} \tau'$ acts trivially on $\mathbb{K}[u] \subseteq \mathbb{L}$.

So $\tau^{-1} \tau' \in Gal(\mathbb{E}/\mathbb{L})$, so $[\tau] = [\tau']$.

$\square$.

<u>Lemma:</u> For any $\mathbb{F} \subset \mathbb{E}$ w/ Galois gp G,

$$
\begin{array}{ccc}
\mathbb{E}^G & & G \\
\cap & & \cup \\
\mathbb{E}^J & \longleftarrow\!\shortmid & J \\
\cap & & \cup \\
\mathbb{E}^H & & H \\
\cap & & \cup \\
\mathbb{E} & & 1
\end{array}
\qquad
[\mathbb{E}^H : \mathbb{E}^J] \leq [J : H].
$$

For both lemmas, going across Galois connection drops the index.

Cor: • If $\mathbb{F} \subset \mathbb{K} \subset \mathbb{L} \subset \mathbb{E}$ and $\mathbb{K}$ is closed and $\mathbb{K} \subset \mathbb{L}$ finite index, then $\mathbb{L}$ is closed.

• If $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F}) \supset J \supset H \supset 1$ and H closed and $H \subset J$ finite index then J closed.

Pf idea: $\mathbb{K} \rightsquigarrow \mathbb{K}$.

$\mathbb{L} \rightsquigarrow \;\geq \mathbb{L}$ because Galois connection.

$\leq \mathbb{L}$ because lower index of $\mathbb{K} \subset \mathbb{L}$. $\square$

<u>Cor:</u> ① and ② in the fundamental thm!

Given $\mathbb{F} \subset \mathbb{K} \subset \mathbb{E}$, $\mathbb{K}$ is <u>stable</u> (in $\mathbb{F} \subset \mathbb{E}$)

if $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ preserves $\mathbb{K}$ as a set.

i.e. $\forall\ \tau \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$, $\tau(\mathbb{K}) = \mathbb{K}$.

From definition unpacking:

<u>Lemma:</u> • If $\mathbb{K}$ stable then $\mathrm{Gal}(\mathbb{E}/\mathbb{K}) \subset \underbrace{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}_{}$ "G" is normal.

• If $J \subset G$ is normal, then $\mathbb{F} \subset \mathbb{E}^J \subset \mathbb{E}$ is stable.

Also defn unpacking:

If $\mathbb{F} \subset \mathbb{K} \subset \mathbb{E}$ and $\mathbb{F} \subset \mathbb{E}$ Galois,

and if $\mathbb{K}$ stable,

then $\mathbb{F} \subset \mathbb{K}$ is Galois.

Idea: If $\mathbb{K}$ stable, There is a map

$$\mathrm{Gal}(\mathbb{E}/\mathbb{F}) \longrightarrow \mathrm{Gal}(\mathbb{K}/\mathbb{F})$$

namely "restrict to $\mathbb{K}$".