The Galois Theory of Graphs

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The paper, "Galois Theory, Graphs and Free Groups", by Brent Everitt, focuses on the connection between graphs, free groups, and their subgroups. This subject has a rich history that can be roughly split into two categories: combinatorial, where graphs give a convienient way of visualizing certain aspects of the free group theory; and topological, where graphs and their mappings are treated as topological objects. This second viewpoint has its origins at the very beginning of combinatorial group theory, and is where this paper draws its philosophy.

The primary focus of this paper is to use the theory of coverings of arbitrary graphs in order to formulate the well known Galois theory of graphs, and extend this to focus on the graph-theoretic implications of finitely generated subgroups.

To begin, we need a precise definition of a graph:

Definition: A graph is a set Γ with an involutary map $^{-1} : \Gamma \to \Gamma$ and an idempotent map $s : \Gamma \to V_\Gamma$, where V_Γ is the set of fixed points of $^{-1}$. Thus a graph has vertices V_Γ , and edges $E_\Gamma := \Gamma/V_\Gamma$ with (i). $s(v) = v$ for all $v \in V_\Gamma$; (ii). $v^{-1} = v$ for all $v \in V_\Gamma$, $e^{-1} \in E_\Gamma$ and $e^{-1} \neq e = (e^{-1})^{-1}$ for all $e \in E_\Gamma$.

While this definition is less elegant than the standard "ordered pair definition," $\Gamma = (V, E)$, this more precise, topological definition is useful in proving results relating to the topology of graphs and their coverings.

A *path* is a finite sequence of mutually incident edges. We have *closed* paths if the first and last vertex are the same, and *trivial* paths consisting of a single vertex. Γ is connected if any two vertices can be joined by a path.

A *map* of graphs is a set map $g : \Gamma \to \Lambda$ with $g(V_\Gamma) \subseteq V_\Lambda$, such that the following diagram commutes, where σ_Γ is one of the s or $^{-1}$ maps for Γ, and σ_Λ similarly.

$$\begin{tikzcd}
\Gamma \arrow[r, "g"] \arrow[d, "\sigma_\Gamma"'] & \Lambda \arrow[d, "\sigma_\Lambda"] \\
\Gamma \arrow[r, "g"] & \Lambda
\end{tikzcd}$$

A map is *dimension preserving* if $g(E_\Gamma) \subseteq E_\Lambda$.
A map $g : \Gamma \to \Lambda$ is a *homeomorphism* if it is dimension preserving and is a bijection on the vertex and edge sets.
A *quotient relation* is an equivalence relation $\sim$ on Γ such that
(i). $x \sim y \Rightarrow s(x) \sim s(y)$ and $x^{-1} \sim y^{-1}$, (ii). $x \sim x^{-1} \Rightarrow [x] \cap V_\Gamma \neq \emptyset$, where $[x]$ is the equivalence class under $\sim$ of x.

Definition: A map $p : \Lambda \to \Delta$ of graphs is a covering iff:
(i). p preserves dimension; and
(ii). for every vertex $v \in \Lambda$, p is a bijection from the set of edges in Λ with start vertex v to the set of edges in Δ with start vertex $p(v)$.

Example: The following figure demonstrates a graph and a covering of the graph:

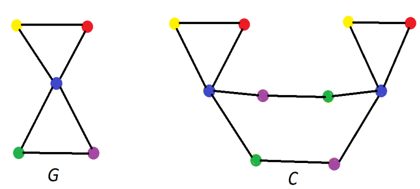

Figure 1: The graph C covers G, with the covering mapping the vertices of C to the same coloured vertex in G.

For our purposes, we will only be concerned with connected graphs, because otherwise we would simply consider the connected components of the graph and apply the results to each component separately. In this case, we can think of a covering intuitively, as a surjective map that preserves neighbourhoods. ie, if there is an edge between vertices x and y in graph Λ, then there must be an edge between vertices $p(x)$ and $p(y)$ in Δ. This will be formalized below.
If $p(x) = y$, then we say that x *covers* y, and y *lifts to* x.
The set of all cells covering y, is its *fiber*, $\text{fib}_{\Lambda \to \Delta}(y)$.
The first useful result that will be helpful in proving the Galois correspondence of graphs is the following:

Lemma A: Let $p : \Lambda \to \Delta$ be a covering.
(i). If Δ is connected then p maps the cells of Λ surjectively onto the cells of Δ.

(ii). If Λ is connected, then the fibers of any two cells of Δ have the same cardinality, called the degree, $deg(\Lambda \to \Delta)$, of the covering.

(iii). If Λ, Δ are connected and $deg(\Lambda \to \Delta) = 1$, then the covering $\Lambda \to \Delta$ is a homeomorphism.

The proof of these statements follow immediately from definition unpacking.

We can now introduce the concept of the lattice structure of graph coverings, which resembles the lattice structures of field extensions:

For connected graphs Λ, Δ, and covering $p : \Lambda_u \to \Delta_v$, two intermediate coverings, $\Lambda_u \to \Gamma_x \to \Delta_v$, and $\Lambda_u \to \Upsilon_y \to \Delta_v$, are *equivalent* if and only if there exists a homomorphism $\beta : \Gamma_x \to \Upsilon_y$ such that the following diagram commutes.

$$\begin{tikzcd}
\Lambda_u \arrow[r] \arrow[dr] & \Gamma_x \arrow[r] \arrow[d] & \Delta_v \\
& \Upsilon_y \arrow[ur] &
\end{tikzcd}$$

Let $\mathcal{L}(\Lambda_u, \Delta_v)$ be the set of equivalence classes of such connected intermediate coverings.

An *automorphism* of p is a graph homeomorphism, $\alpha : \Lambda_u \to \Lambda_{u'}$, making the following diagram commute:

$$\begin{tikzcd}
\Lambda_u \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & \Lambda_{u'} \arrow[dl, "p"] \\
& \Delta_v &
\end{tikzcd}$$

Definition: The set of automorphisms of $p : \Lambda_u \to \Delta_v$, denoted $\mathrm{Gal}(\Lambda_u, \Delta_v)$, is called the *Galois Group* of p.

If $\mathrm{Gal}(\Lambda_u, \Delta_v)$ acts regularly on $fib_{\Lambda \to \Delta}(v)$, then it is called a "Galois covering".

Lemma B: If $H_1 \subset H_2 \subset \mathrm{Gal}(\Lambda_u, \Delta_v)$, then,

$$\Lambda_u \xrightarrow{q_1} (\Lambda/H_1)_{q_1(u)} \xrightarrow{s} (\Lambda/H_2)_{q_2(u)} \xrightarrow{r} \Delta_v$$

are all coverings, where $q_i : \Lambda \to \Lambda/H_i$ are the quotient maps and s, r are defined by $sq_1 = q_2$ and $rq_2 = p$.
This follows from the fact that the H_i's act freely and preserve orientation on the graphs, giving that the q_i's are coverings. Since q_1 and $q_2 = sq_1$ are coverings, then s is also a covering, and likewise for r.

Proposition: Let $p : \Lambda_u \to \Delta_v$ be a Galois covering. If $H \subset \text{Gal}(\Lambda_u, \Delta_v)$, then $[\text{Gal}(\Lambda_u, \Delta_v) : H] = \deg(\Lambda/H_{q(u)} \xrightarrow{r} \Delta_v)$.
Proof: If a group G acts regularly on a set and H is a subgroup, then the number of H-orbits is the index $[G : H]$. The result follows as the H-orbits on the fiber (via p) of v are precisely the vertices of Λ/H covering v (via r).

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The Galois thoery of graphs is concerned with relating the relationship between these graph coverings, (or more precisely, the intermediate coverings), with the groups of automorphisms generated by a covering. This relationship forms an anti-isomorphism, exactly as in the Galois theory of field extensions. The Galois correspondence of graphs is as follows:

Theorem: Let $\Lambda_u \to \Delta_v$ be a Galois covering with $\mathcal{L}(\Lambda_u, \Delta_v)$ the lattice of equivalence classes of intermediate coverings and $G = \text{Gal}(\Lambda_u, \Delta_v)$ the Galois group. Then the map that associates $\Lambda_u \to \Gamma_x \to \Delta_v \in \mathcal{L}(\Lambda_u, \Delta_v)$, to the subgroup $\text{Gal}(\Lambda_u, \Gamma_x)$ is a lattice anti-isomorphism from $\mathcal{L}(\Lambda_u, \Delta_v)$ to the lattice of subgroups of $\text{Gal}(\Lambda_u, \Delta_v)$. Its inverse is the map associating $H \subset \text{Gal}(\Lambda_u, \Delta_v)$ to the element $\Lambda_u \to \Lambda/H_{q(u)} \to \Delta_v \in \mathcal{L}(\Lambda_u, \Delta_v)$.

Proof: Let f and g be the maps described in the statement of the theorem, f : intermediate coverings $\to$ subgroups of G, g : subgroups of $G \to$ intermediate coverings.
If $H_1 \leq H_2$ in the subgroup lattice, then by Lemma B we see that $g(H_2) \leq g(H_1)$, so g is an anti-morphism of lattices.
If $\Lambda_u \to \Gamma_x \to \Delta_v$ is an intermediate covering, then we also have the intermediate covering $\Lambda_u \to \Lambda_u/\text{Gal}(\Lambda_u, \Gamma_x) \to \Gamma_x$ with $\Lambda_u \to \Gamma_x$ is Galois. By our Proposition, we see that the covering $\Lambda_u/\text{Gal}(\Lambda_u, \Gamma_x) \to \Gamma_x$ must have degree 1, and is therefore a homeomorphism by Lemma A(iii). From this, we have the following diagram:

$$\begin{tikzcd}
 & \Lambda_u/\mathrm{Gal}(\Lambda_u, \Gamma_x) \arrow[dd] \arrow[dr] & \\
\Lambda_u \arrow[ur] \arrow[dr] & & \Delta_v \\
 & \Gamma_x \arrow[ur] &
\end{tikzcd}$$

Notice that the whole square, as well as the left triangle, commute simply by intermediacy, hence the right triangle commutes as well. Therefore the intermediate coverings $\Lambda_u \to \Gamma_x \to \Delta_v$ and $\Lambda_u \to \Lambda_u/\mathrm{Gal}(\Lambda_u, \Gamma_x) \to \Delta_v$ are actually equivalent, and we have that gf =id.
If $H \subset \mathrm{Gal}(\Lambda_u, \Delta_v)$ and $q : \Lambda \to \Lambda/H$ is the quotient map, then $q\alpha = q$ for any $\alpha \in H$, which gives $H \subset \mathrm{Gal}(\lambda_u, (\Lambda/H)_{q(u)})$ with the covering $\Lambda_u \to \Lambda/H_{q(u)}$ being an intermediate covering, and hence a Galois covering. Our Proposition gives that the index of H in $\mathrm{Gal}(\Lambda_u, (\Lambda/H)_{q(u)})$ to be the degree of the covering $\Lambda/H_{q(u)} \to \Lambda/h_{q(u)}$, which is trivially 1.
Therefore, $H = \mathrm{Gal}(\Lambda_u, (\Lambda/H)_{q(u)})$, and we get that fg =id.

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Everitt concludes this section of the paper by noting that, as a special case of the above theorem, we can find an even more familiar version of the Galois correspondence if we restrict our attention to what is known as the "universal cover" of a graph.
These resuts have since been extended from graphs, or combinatorial 1-complexes, to combinatorial 2-complexes by using Cayley graphs in order to prove results on solvable groups. This was done by Fatemeh Ghanei and Hanieh Mirebrahimi in their paper "Solvable groups, Cayley graphs and Complexes".

References:

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