Math 5055: Advanced Algebra II

Assignment 1

due January 24, 2022

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. Show that $x^3 + 9x + 6$ is irreducible over $\mathbb{Q}$. Let θ be a root, and compute $(1 + \theta)^{-1} \in \mathbb{Q}[\theta]$.

2. Show that $x^3 + x + 1$ is irreducible over $\mathbb{F}_2$. Let θ be a root, and compute its powers in $\mathbb{F}_2[\theta]$.

3. Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two finite extensions of a field $\mathbb{F}$, both subextensions of a common extension $\mathbb{E}$; recall that $\mathbb{K}_1\mathbb{K}_2 \subset \mathbb{E}$ is the subextension that they generate. Show that the tensor product algebra $\mathbb{K}_1 \otimes_{\mathbb{F}} \mathbb{K}_2$ is a field if and only if $[\mathbb{K}_1\mathbb{K}_2 : \mathbb{F}] = [\mathbb{K}_1 : \mathbb{F}][\mathbb{K}_2 : \mathbb{F}]$. Conclude that this happens in particular whenever $[\mathbb{K}_1 : \mathbb{F}]$ and $[\mathbb{K}_2 : \mathbb{F}]$ are coprime.

4. In the field $\mathbb{F}(x)$ of rational functions, let $u = x^3/(x + 1)$, and consider the subfield $\mathbb{F}(u) \subset \mathbb{F}(x)$. Compute the degree of this field extension.

   Hint: $\mathbb{F}(x) = \mathbb{F}(u)(x)$. Show that x is algebraic over $\mathbb{F}(u)$, and find its minimal polynomial.

5. A field $\mathbb{F}$ is *formally real* if -1 is not a sum of squares in $\mathbb{F}$. Suppose that $\mathbb{F}$ is formally real and that $f(x) \in \mathbb{F}[x]$ is irreducible of odd degree, and pick a root α of $f(x)$. Show that $\mathbb{F}(\alpha)$ is formally real.

   Hint: Consider a counterexample of minimal degree. Show that there exists $g(x)$ of odd degree $< \deg(f)$ such that $-1 + f(x)g(x)$ is a sum of squares in $\mathbb{F}[x]$. Show that $g(x)$ would give a new counterexample, violating the minimality of f.

6. Let $\mathbb{E}$ be a finite extension of $\mathbb{F}$. Show that $\mathbb{E}$ is a splitting field (of some set of polynomials) over $\mathbb{F}$ if and only if every irreducible polynomial over $\mathbb{F}$ which admits a root in $\mathbb{E}$ splits completely in $\mathbb{E}$.

   Hint: Use that if $\phi : \mathbb{F} \xrightarrow{\sim} \mathbb{F}'$ is a field isomorphism, and $\mathbb{F} \subset \mathbb{E}$ and $\mathbb{F}' \subset \mathbb{E}'$ are splitting fields for "the same" polynomials (i.e. corresponding via ϕ), then ϕ extends to an isomorphism $\mathbb{E} \xrightarrow{\sim} \mathbb{E}'$.