Math 5055: Advanced Algebra II

Assignment 3

due the first week in March

Homework should be submitted either as a single PDF attachment to `theojf@dal.ca` (please include your name in the file name!) or as a single stapled(!) collection to my mailbox in the Chase building.

Noncommutative (in)separability

Let $\mathbb{K}$ be a field and A an associative and unital, but potentially noncommutative, $\mathbb{K}$-algebra. Recall that the multiplication map $m : A \otimes A \to A$ is a map of A-bimodules. In other words, suppose that $u = \sum_i u_i^{(1)} \otimes u_i^{(2)} \in A \otimes A$, so that $m(u) = \sum_i u_i^{(1)} u_i^{(2)} \in A$. Then given $a_1, a_2 \in A$, we have:

$$m\big((a_1 \otimes 1)u(1 \otimes a_2)\big) = a_1\, m(u)\, a_2.$$

1. Confirm that m is an algebra homomorphism if and only if A is commutative.

The algebra A is called *separable* if multiplication splits as a map of A-bimodules: there should be a linear $\Delta : A \to A \otimes A$ such that $\Delta(a_1 b a_2) = (a_1 \otimes 1)\Delta(b)(1 \otimes a_2)$ for all $a_1, b, a_2 \in A$, and such that $m \circ \Delta = \mathrm{id}_A$. A choice of Δ will be called a *separation* for A.

2. Prove that every separation Δ is *coassociative*:

   $$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta : A \to A \otimes A \otimes A.$$

   Remark: The name is because the dual map $\Delta^* : A^* \otimes A^* \to (A \otimes A)^* \to A^*$ is associative.

   Hint: Precompose both sides with $\mathrm{id}_A = m \circ \Delta$.

3. Confirm that the data of a separation Δ is equivalent to a choice of $u \in A \otimes A$ such that $m(u) = 1$ and $(a \otimes 1)u = u(1 \otimes a)$ for all $a \in A$.

Suppose A is finite-dimensional. A is *semisimple* if every finite-dimensional left A-module is projective.

4. Show that if A is finite-dimensional and separable, then it is semisimple. Explain why Question 6 below shows that the converse does not hold.

   Hint: Let M be a left A-module. Find M as a direct summand of the free left A-module $A \otimes_{\mathbb{K}} M \cong A^{\dim M}$. **Hint:** Use the isomorphism $A \otimes_A M \cong M$ of left A-modules.

5. Suppose that A is finite-dimensional, and let A^{op} denote its opposite algebra. Show that A is separable if and only if $A^e := A \otimes A^{\mathrm{op}}$ is semisimple.

Hint: In one direction, show that the A is separable if and only if A^{op} is separable, and that the tensor product of separable algebras is separable. In the other direction, write A as a left A^e-module.

Remark: The algebra $A^e := A \otimes A^{\mathrm{op}}$ is sometimes called the *enveloping algebra* of A.

6. Let $\mathbb{K} \subset \mathbb{L}$ be a finite-degree field extension which is *inseparable* in the sense of fields. Show that it is inseparable in the sense of algebras.

   Hint: Find an element $x \in \mathbb{L}$ such that $x^p \in \mathbb{K}$ but $x \notin \mathbb{K}$. Let $I \subset \mathbb{L} \otimes \mathbb{L}$ be the ideal generated by $y := x \otimes 1 - 1 \otimes x$, and consider the quotient module $(\mathbb{L} \otimes \mathbb{L})/I$.

   Remark: Using Galois theory, we showed in class that if $\mathbb{L}$ is separable in the sense of fields, then it is separable in the sense of algebras.

Commutative (in)separability

Recall that a field $\mathbb{K}$ is *perfect* if every algebraic extension of $\mathbb{K}$ is separable. For example we showed that fields of characteristic 0 are perfect and finite fields are perfect.

7. Show that $\mathbb{K}$ is perfect of positive characteristic if and only if the Frobenius endomorphism of $\mathbb{K}$ is an isomorphism.

8. Suppose that $\mathbb{K} \subset \mathbb{L}$ is an extension of index $n < \infty$, and suppose that n is not divisible by the characteristic of $\mathbb{K}$. Prove that $\mathbb{L}$ is separable over $\mathbb{K}$.

9. Let $\mathbb{K} \subset \mathbb{L}$ be an algebraic extension of fields. Suppose that $u \neq 0 \in \mathbb{L}$ is separable over $\mathbb{K}$ and $v \neq 0 \in \mathbb{L}$ is purely inseparable over $\mathbb{K}$. Prove that $\mathbb{K}(u, v) = \mathbb{K}(u + v) = \mathbb{K}(uv)$.

10. Suppose that $\mathbb{K}$ has characteristic $p > 0$, and that $\mathbb{L} = \mathbb{K}(u, v)$ is an extension of $\mathbb{K}$ of degree p^2. Suppose further that $u^p, v^p \in \mathbb{K}$. Show that $\mathbb{L} \neq \mathbb{K}(w)$ for any single element w, and exhibit infinitely many subextensions of $\mathbb{K} \subset \mathbb{L}$.

11. Let $\mathbb{K} \subset \mathbb{L}$ be an algebraic extension, and let $\mathbb{L}_s \subset \mathbb{L}$ be the subfield of separable (over $\mathbb{K}$) elements. Let $\bar{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$ and $\mathbb{K}^s \subset \bar{\mathbb{K}}$ the separable closure (i.e. $\mathbb{K}^s = \bar{\mathbb{K}}_s$). Show that the sets $\hom(\mathbb{L}, \bar{\mathbb{K}})$ and $\hom(\mathbb{L}_s, \mathbb{K}^s)$ are canonically isomorphic (where the hom is in the category of commutative $\mathbb{K}$-algebras). Conclude that $\hom(\mathbb{L}, \bar{\mathbb{K}})$ is of cardinality equal to the index $[\mathbb{L} : \mathbb{K}]_s := [\mathbb{L}_s : \mathbb{K}]$.

    Remark: This index is called the *separable degree* of $\mathbb{K} \subset \mathbb{L}$.

12. Suppose that $\mathbb{L} = \mathbb{K}(u)$ is a simple algebraic extension. What is $[\mathbb{L} : \mathbb{K}]_s$ in terms of the minimal polynomial of u?