Math 5055: Advanced Algebra II

Assignment 4

due next week

Galois groups

1. Determine the Galois groups of the following polynomials over the fields indicated.
   (a) $x^4 - 5$ over $\mathbb{Q}$; over $\mathbb{Q}[\sqrt{5}]$; over $\mathbb{Q}[\sqrt{-5}]$.
   (b) $x^3 - x - 1$ over $\mathbb{Q}$; over $\mathbb{Q}[\sqrt{-23}]$.
   (c) $x^4 + 3x^3 + 3x - 2$ over $\mathbb{Q}$.
   (d) $x^5 - 6x + 3$ over $\mathbb{Q}$.

2. Which roots of unity are contained in the following fields?
$$\mathbb{Q}[\sqrt{-1}], \quad \mathbb{Q}[\sqrt{2}], \quad \mathbb{Q}[\sqrt{3}], \quad \mathbb{Q}[\sqrt{5}], \quad \mathbb{Q}[\sqrt{-2}], \quad \mathbb{Q}[\sqrt{-3}].$$

3. Let $\mathbb{K} \subset \mathbb{L}$ be an extension of finite fields, i.e. $\mathbb{K} = \mathbb{F}_q$ and $\mathbb{L} = \mathbb{F}_{q^m}$ for some prime power $q = p^n$. Show that both the Trace $T_{\mathbb{K}}^{\mathbb{L}} : \mathbb{L} \to \mathbb{K}$ and the Norm $N_{\mathbb{K}}^{\mathbb{L}} : \mathbb{L} \to \mathbb{K}$ are surjective.

Representations of finite groups

4. Given a group G, let $G' < G$ denote its *commutator subgroup*, i.e. the subgroup generated by elements of the form $ghg^{-1}h^{-1}$.
   (a) Show that G' is normal in G, and that the quotient G/G' is abelian.
   (b) Show that the one-dimensional representations of G are in bijection with the one-dimensional representations of G/G'.

5. Suppose that A is a finite abelian group and $\mathbb{F}$ is algebraically closed. Show that a representation of A over $\mathbb{F}$ is irreducible if and only if it is 1-dimensional.

6. Suppose that G is a group, with centre $Z(G)$. Suppose that $(V, \rho_V : G \to \mathrm{GL}(V))$ is an irreducible representation of G over an algebraically closed field $\mathbb{F}$. Show that if $c \in Z(G)$ then $\rho_V(c)$ is a scalar multiple of the identity operator. In other words, show that ρ restricts to a homomorphism $Z(G) \to \mathbb{F}^\times$. This homomorphism is called the *central character* of the representation (V, ρ_V).

   Conclude that, if a finite group G admits a faithful irreducible representation, then its centre must be cyclic.

7. Let p be a positive prime, P a p-group (i.e. a group of order a power of p), and $\mathbb{F}$ a field of characteristic p. Prove that the only irreducible representation of P over $\mathbb{F}$ is the trivial one.

   Hints: P contains a central element c of order p. Use exercise 6.