Math 5055: Advanced Algebra II

Assignment 5

Solutions

1. **Show that the character table of a product of finite groups is the tensor product of their character tables.**

   Suppose that G_1, G_2 are finite groups and I_1, I_2 are irreps thereof. Then $I_1 \otimes I_2$ is naturally a representation of $G_1 \times G_2$. The character is

   $$\chi_{I_1 \otimes I_2}((g_1, g_2)) = \operatorname{tr}_{I_1 \otimes I_2}(g_1 \otimes g_2) = \operatorname{tr}_{I_1}(g_1) \operatorname{tr}_{I_2}(g_2) = \chi_{I_1}(g_1)\chi_{I_2}(g_2).$$

   We claim that $I_1 \otimes I_2$ is irreducible, and that different choices for I_1, I_2 lead to different irreps. To show this, it suffices to show that

   $$\langle \chi_{I_1 \otimes I_2}, \chi_{J_1 \otimes J_2} \rangle = \begin{cases} 1, & I_1 \cong J_1 \text{ and } I_2 \cong J_2 \\ 0, & \text{otherwise.} \end{cases}$$

   But note that the conjugacy classes in $G_1 \times G_2$ are precisely the products of conjugacy classes (in other words, (g_1, g_2) is conjugate to (h_1, h_2) if and only if g_1 is conjugate to h_1 and g_2 is conjugate to h_2). From this one finds that

   $$\langle \chi_{I_1 \otimes I_2}, \chi_{J_1 \otimes J_2} \rangle = \langle \chi_{I_1}, \chi_{J_1} \rangle \langle \chi_{I_2}, \chi_{J_2} \rangle.$$

   So the irreps of $G_1 \times G_2$ are precisely the tensor products of irreps. In other words, the rows of the character table of $G_1 \times G_2$ are indexed by pairs of rows from the two character tables. We already remarked that the conjugacy classes in $G_1 \times G_2$ are precisely the products of conjugacy classes; in other words, the columns of the character table of $G_1 \times G_2$ are indexed by pairs of columns from the two character tables. And the formula $\chi_{I_1 \otimes I_2}((g_1, g_2)) = \chi_{I_1}(g_1)\chi_{I_2}(g_2)$ shows that the entries are the products of entries.

   This is precisely the definition of tensor product of matrices.

2. **Calculate the character table of the alternating group A_5.**

   An element of A_5 is either trivial, a 3-cycle, a 5-cycle, or a product of two 2-cycles. Any two 3-cycles are conjugate: indeed, to conjugate $(123)(4)(5)$ to $(abc)(d)(e)$, we could use either of the two permutations $[12345] \mapsto [abcde]$ or $[12345] \mapsto [abced]$, and one of these is an even permutation. Similarly, any two products of two 2-cycles are conjugate. For example, to take $(12)(34)(5)$ to $(ab)(cd)(e)$, we could use either $[12345] \mapsto [abcde]$ or $[12345] \mapsto [abdce]$, and one of these is even. But there are two conjugacy classes of 5-cycles. To see this, note that a 5-cycle is always of the form $(1abcd)$ for a unique permutation $[abcd]$ of $[2345]$, and that permutation is either even or odd. If it is even, then $(1abcd)$ is conjugate to (12345) via said even permutation. So it suffices to show that (12345) is not conjugate to (12354)

by an even permutation. Well, any the permutations conjugating (12345) to (12354) are $[12345] \mapsto [12354], [23541], [35412], [54123], [41235]$, and these are all odd.

Choosing representatives for our conjugacy classes, and immediately recording the trivial representation, we have:

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_1	1	1	1	1	1

There are four more irreducible characters.

A_5 is simple, so in particular there are no other 1D representations. To make progress, we should look for some other interesting representations of A_5. Well, we know one of them: the permutation representation, with character

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_{perm}	5	2	1	0	0

We calculate

$$\|\chi_{\text{perm}}\|^2 = \frac{1}{\#G} \sum_{[g]} \#[g] \cdot \chi_{\text{perm}}(g)^2 = \frac{1}{60}(1 \cdot 5^2 + 20 \cdot 2^2 + 15 \cdot 1^2 + 12 \cdot 0^2 + 12 \cdot 0^2) = 2.$$

So χ_{perm} is a sum of two inequivalent irreps. One of them is the trivial representation, and the other is $\chi_2 = \chi_{\text{perm}} - \chi_1$.

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_2	4	1	0	-1	-1

The 24 elements of order 5 in A_5 sort themselves into 6 Sylow 5-subgroups. (Each one contains two elements from each of the two conjugacy classes.) The conjugation action of A_5 on that set of size 6 has character ψ given by

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
ψ	6	0	2	1	1

with

$$\|\psi\|^2 = \frac{1}{60}(1 \cdot 6^2 + 15 \cdot 2^2 + 12 \cdot 1^2 + 12 \cdot 1^2) = 2.$$

Since this is a permutation representation, it contains a trivial subrep, and we find that $\chi_3 = \psi - \chi_1$ is irreducible.

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_3	5	-1	1	0	0

We need two more irreps, χ_4 and χ_5. Note that $60 = 1^2 + 4^2 + 5^2 + \chi_4(1)^2 + \chi_5(1)^2$. And $\chi(1)$ is a dimension hence a nonnegative integer, so $\chi_4(1) = \chi_5(1) = 3$. In other words, there are two 3-dimensional irreps of A_5.

We can work out what $\chi_4 + \chi_5$ is from using the fact that

$$\chi_{\mathbb{C}A_5} = \chi_1 + 4\chi_2 + 5\chi_3 + 3\chi_4 + 3\chi_5,$$

or in other words $\chi_4 + \chi_5 = \frac{1}{3}(\chi_{\mathbb{C}A_5} - \chi_1 - 4\chi_2 - 5\chi_3)$. We find:

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
$\chi_4 + \chi_5$	6	0	-2	1	1

Let's suppose that one of these characters, χ_4, say, is

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_4	3	x	y	z	w

Then χ_4 is orthogonal to each of χ_1, χ_2, χ_3, and hence with any linear combination thereof. Orthogonality with χ_3 gives

$$0 = 15 + 20 \cdot -x + 15 \cdot y,$$

whereas orthogonality with $\chi_1 + \chi_2$ gives

$$0 = 15 + 20 \cdot 2x + 15 \cdot y.$$

Together these equations imply $x = 0$ and $y = -1$. Now orthogonality of χ_4 with χ_1 gives

$$0 = 3 + 20 \cdot 0 + 15 \cdot -1 + 12 \cdot z + 12 \cdot w$$

and so $z + w = 1$.

In other words, our character table so far looks like

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	z	$1-z$
χ_5	3	0	-1	$1-z$	z

Note that $(12345)^{-1} = (15432)$ is conjugate to (12345). It follows that all characters are real. Then orthogonality of χ_4 and χ_5 gives the equation

$$0 = 9 + 15 + 24z(1-z).$$

Writing $\phi = \frac{1+\sqrt{5}}{2}$ for the Golden Ratio, we find that $z = \phi$ or $\phi^{-1} = 1 - \phi$. Thus the final answer is

class	1	(123)	(12)(34)	(12345)	(12354)
$\#[g]$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	ϕ	ϕ^{-1}
χ_5	3	0	-1	ϕ^{-1}	ϕ

Remark: Note that χ_4 and χ_5 are Galois conjugate. They are also obviously exchanged by the outer automorphism of A_5 given by conjugating by the odd permutation (1)(2)(3)(45). So for all intents and purposes they are "the same" — if you and I each have a group abstractly isomorphic to A_5, there's no way to decide which irrep is χ_4 and which is χ_5.

What actually is this 3-dimensional irrep?

Consider the regular dodecahedron. Let G denote its group of rotational symmetries. Then $\#G = 60$: you can rotate any face to any other face (12 choices) and then rotate that face (5 choices). There are five ways to inscribe a cube into a dodecahedron, and the 60 rotational symmetries transitively permute those five inscribed cubes. It is not too hard to show directly that the map $G \to S_5$ given by permuting those five cubes is actually an isomorphism $G \cong A_5$.

On the other hand, by construction G acts on the $\mathbb{R}^3$ containing the dodecahedron. So this affords a 3D representation of $G \cong A_5$, and it turns out to be either χ_4 or χ_5 depending on which ordering of the cubes you use to build your isomorphism.

3. **Let G be a finite group of order n.**

   (a) **Show that the function $g \mapsto g^m$ is a bijection on G if and only if m is coprime to n, and that it only depends on the value of m modulo n.**

   The order of every element divides n. If m and n share a prime p as a common factor, then $g \mapsto g^m$ will take all elements of order p to 1; there does exist an element of order p, and so that map is not a bijection. If m and n are coprime, then we can solve $mm' = 1 \pmod{n}$, and $g \mapsto g^m$ will be inverse to the map $g \mapsto g^{m'}$.

   (b) **Suppose that m is coprime to n. Give an example to show that $g \mapsto g^m$ is typically not a group homomorphism. Nevertheless, show that g^m and g always have the same order.**

   For example, -1 is coprime to every number. But $(gh)^{-1} = h^{-1}g^{-1} \neq g^{-1}h^{-1}$ unless g and h commute.

   If g has order o, then $(g^m)^o = g^{mo} = g^{om} = 1^m = 1$, so g^m has order dividing o. But by using the explicit inverse map from (a), we see that o divides the order of g^m.

   (c) **Show that the character table of G takes values in the cyclotomic field $\mathbb{Q}(\xi)$, where $\xi = \sqrt[n]{1}$ is a primitive nth root of unity.**

   If χ is a character and $g \in G$, then $\chi(g)$ is a sum of eigenvalues of g. Since g has order dividing n, these eigenvalues are all powers of ξ.

   (d) **How does the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ relate to the set of numbers m such that m is coprime to n?**

   $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) = (\mathbb{Z}/n)^\times$ is precisely the set of classes modulo n of numbers m such that m is coprime to n. The class $m \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ acts by $\xi \mapsto \xi^m$.

   (e) **Let χ be a character of G. How does $\chi(g^m)$ relate to $\chi(g)$?**

   The eigenvalues of g^m are the mth powers of eigenvalues of g. In other words, the images of said eigenvalues under the Galois automorphism $m \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. So $\chi(g^m)$ is the image of $\chi(g)$ under that Galois automorphism.

   (f) **Let $g \in G$ be an arbitrary element. Show that the following statements are equivalent:**

      i. $\chi(g) \in \mathbb{Q}$ **for every character** χ.

      ii. $\chi(g) \in \mathbb{Z}$ **for every character** χ.

iii. g is conjugate to g^m for every number m coprime to n.

i. and ii. are equivalent $\chi(g)$ is a sum of roots of unity and hence an algebraic integer.

Suppose that g is conjugate to g^m. Then $\chi(g) = \chi(g^m)$ for every character χ, but the latter is a Galois conjugate of the former. Since the extension $\mathbb{Q} \subset \mathbb{Q}(\xi)$ is Galois, we learn that $\chi(g) \in \mathbb{Q}$. Thus iii. implies i.

Conversely, if $\chi(g) \in \mathbb{Q}$, then $\chi(g) = \chi(g^m)$. If this holds for every character χ, then the conjugacy classes of g and g^m must be equal, since not just the rows but also the columns of the character table are orthogonal. Thus i. implies iii.

(g) **Show that condition iii above, and hence also the other two conditions, holds for example when $G = S_n$ is a symmetric group.**

The order of S_n is $n!$. If m is coprime to $n!$, then in particular it is coprime to every $k \leq n$. It follows that for any permutation $g \in S_n$, the cycle structures of g and g^m are the same. But conjugacy classes in S_n are determined by cycle structures.

4. **Let p be an odd prime, and P a nonabelian group of order p^3. Describe the character table of P, and show that it does not depend on which group you use.**

We will use the following:

Lemma: Let G be a nonabelian group, with centre $Z(G)$. Then $G/Z(G)$ is not cyclic.

Proof: Suppose that $G/Z(G)$ is cyclic of order n, and choose an element $x \in G$ that maps to a generator of $G/Z(G)$. Then the elements of G are all of the form zx^i for $i = 1, \dots, n-1$ and $z \in Z(G)$. But these all commute, violating the requirement that G was nonabelian.

Now, the exercise asks about the p-group P. Every p-group has a nontrivial centre. So the quotient $P/Z(P)$ has order either p or p^2, and so is either $\mathbb{Z}/p$ or $\mathbb{Z}/(p^2)$ or $(\mathbb{Z}/p)^2$. The first two are ruled out by the lemma. Thus $Z(P)$ has order p.

Since $(\mathbb{Z}/p)^2$ is abelian, the commutator subgroup P' must be contained in $\ker(P \to (\mathbb{Z}/p)^2) = Z(P) \cong \mathbb{Z}/p$. The commutator subgroup cannot be trivial since P is not itself abelian. So $P' = Z(P) \cong \mathbb{Z}/p$.

Thus the 1D representations of P are precisely the 1D representations of $(\mathbb{Z}/p)^2$. There are p^2 many of these.

To count how many irreps we need, we instead count conjugacy classes. Pick $x, y \in (\mathbb{Z}/p)^2$, lift them to $\tilde{x}, \tilde{y} \in P$, and compute $[\tilde{x}, \tilde{y}] \in Z(P) \cong \mathbb{Z}/p$. Then $[\tilde{x}, \tilde{y}]$ is independent of the choice of lifts and so we will simply write it as $[x, y]$. Moreover, this commutator map $[,] : (\mathbb{Z}/p)^2 \to \mathbb{Z}/p$ is nontrivial and bilinear. It follows that for any $z \in Z(P)$ and $\tilde{x} \in P$ noncentral, there exists $\tilde{y}$ such that $[x, y] = z$. But then $\tilde{y}^{-1}\tilde{x}\tilde{y} = z\tilde{x}$. In other words, for any nontrivial element of $(\mathbb{Z}/p)^2$, all of its lifts are conjugate. A central element is not conjugate to anything other than itself, and so we find that P has: p conjugacy classes of order 1, namely the central elements; $p^2 - 1$ conjugacy classes of order p.

So there are $p^2 + p - 1$ total irreps of P. We've written p^2 of them. For the others, the centre acts nontrivially. By Schur's lemma (see last homework), any central element will act by a scalar. Suppose that a generator z of $Z(P)$ acts by ξ, some nontrivial pth root of unity. If $x \in P$ is noncentral, then $\mathrm{tr}(x) = \mathrm{tr}(zx)$, since x and zx are conjugate, but z is just the scalar matrix ξ, so $\mathrm{tr}(x) = \xi\,\mathrm{tr}(x)$, and so $\mathrm{tr}(x) = 0$.

The last thing to remark is that if we have an irrep where $z \in Z$ takes value ξ, then we have an irrep where z takes value any Galois conjugate of ξ. But the Galois conjugates of ξ are the powers of ξ. So all of these non-one-dimensional characters have the same dimension d. This d must solve $(p-1)d^2 + p^2 = p^3$, or in other words $d = p$.

So the character table of P looks as follows. There are p^2 one-dimensional characters, indexed by pairs $(\alpha, \beta) \in (\mathbb{Z}/p)^2$. These characters take value 1 on all p central elements. The other $p^2 - 1$ conjugacy classes are indexed by pairs $(a, b) \neq (0, 0) \in (\mathbb{Z}/p)^2$, and $\chi_{\alpha,\beta}(a, b) = \exp\bigl(\frac{2\pi i}{p}(\alpha a + \beta b)\bigr)$. Finally, there are $p - 1$ irreducible characters, indexed by $\zeta \neq 0 \in \mathbb{Z}/p$. These characters vanish on noncentral elements in P, and on the central elements $z \in Z(P) \cong \mathbb{Z}/p$, they take the values $\chi_\zeta(z) = d \exp\bigl(\frac{2\pi i}{p}\zeta z\bigr)$.