Math 5055: Advanced Algebra II

Assignment 5

due next week

1. Show that the character table of a product of finite groups is the tensor product of their character tables.

2. Calculate the character table of the alternating group A_5.

3. Let G be a finite group of order n.

   (a) Show that the function $g \mapsto g^m$ is a bijection on G if and only if m is coprime to n, and that it only depends on the value of m modulo n.

   (b) Suppose that m is coprime to n. Give an example to show that $g \mapsto g^m$ is typically not a group homomorphism. Nevertheless, show that g^m and g always have the same order.

   (c) Show that the character table of G takes values in the cyclotomic field $\mathbb{Q}(\xi)$, where $\xi = \sqrt[n]{1}$ is a primitive nth root of unity.

   (d) How does the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ relate to the set of numbers m such that m is coprime to n?

   (e) Let χ be a character of G. How does $\chi(g^m)$ relate to $\chi(g)$?

   (f) Let $g \in G$ be an arbitrary element. Show that the following statements are equivalent:

      i. $\chi(g) \in \mathbb{Q}$ for every character χ.

      ii. $\chi(g) \in \mathbb{Z}$ for every character χ.

      iii. g is conjugate to g^m for every number m coprime to n.

   (g) Show that condition iii above, and hence also the other two conditions, holds for example when $G = S_n$ is a symmetric group.

4. Let p be an odd prime, and P a nonabelian group of order p^3. Describe the character table of P, and show that it does not depend on which group you use.

   Hint: See Exercise 17 of §19.1 of *Dummit and Foote* for an outline.