Galois Connections in Orders, Logic, and Quantum Mechanics

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Galois Connections in Order Theory

Sets With Partial Orderings

DEFINITION [1]: A **partially ordered set (poset)** is a set P together with a binary relation $(\leq) \subseteq P \times P$ such that the following properties are satisfied.

1. **Reflexivity**: for all $x \in P$, $x \leq x$.
2. **Antisymmetry**: for all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. **Transitivity**: for all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

EXAMPLE [1]: The following are partially ordered sets.

1. *The real numbers ordered by the standard less-than-or-equal operation forms a poset* $(\mathbb{R}, \leq)$*.*
2. *If X is an arbitrary set, then* $(\mathcal{P}(X), \subseteq)$ *forms a poset.*

Posets With Lattice Structure

DEFINITION [2]: A **complete lattice** is a poset (P, ≤) such that every subset S⊆P has a least upper bound denoted ⋀S (the **meet** of S) and a greatest lower bound ⋁S (the **join** of S).

EXAMPLE [2]: Do (ℝ, ≤) and (𝒫(X), ⊆) form complete lattices?

1. *(Sketch). We know from analysis that every bounded set* S⊆ℝ *has an infimum (a meet) and a supremum (a join). However, not all sets in* ℝ *are bounded. To correct this we require the extended reals* ℝ ∪ {-∞,∞}.
2. *(Sketch). Let* S ∈ 𝒫(X)*. Then* ⋀S *is intersection and* ⋁S *is union. It follows from set theory that* ⋀S *is a greatest lower bound and* ⋁S *is a least upper bound. This is well defined since* ∅ ∈ 𝒫(X) *and* X ∈ 𝒫(X).

Order-Theoretic Functions on Posets

DEFINITION [3]: Let $(P, \leq)$ and $(Q, \leq)$ be posets, $f: P \to Q$, and $g: P \to P$.

1. f is **isotone** if $x \leq y \Rightarrow f(x) \leq f(y)$ for all $x, y \in P$.
2. f is **antitone** if $x \leq y \Rightarrow f(y) \leq f(x)$ for all $x, y \in P$.
3. f is an (dual) **isomorphism of posets** if f is an isotone (resp. antitone) bijection with an isotone inverse.
4. g is **extensive** if $x \leq g(x)$ for all $x \in P$.
5. g is **idempotent** if $g(g(x)) = g(x)$ for all $x \in P$.

EXAMPLE [3]: The following are examples of adjective-functions between posets.

1. *For every poset, its identity function is an isomorphism of posets.*
2. *The function* $f: x \mapsto 2x$ *is an isomorphism of posets between* $(\mathbb{Z}, \leq)$ *and* $(2\mathbb{Z}, \leq)$.
3. *There exists posets* $(P, \leq)$ *and* $(Q, \leq)$ *with a non-isomorphic isotone bijection.*

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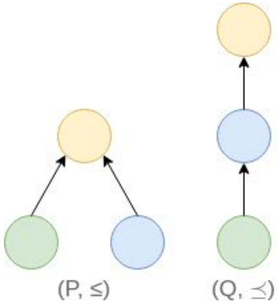
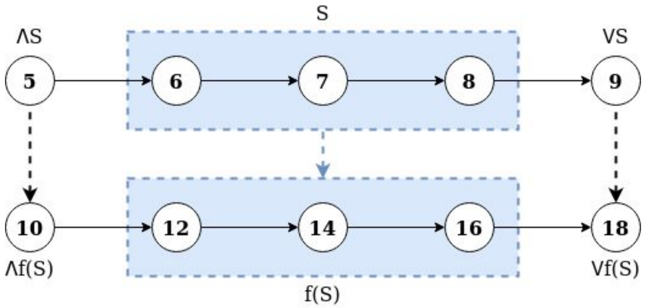


Order-Theoretic Functions on Lattices

The preceding adjectives also apply to functions on lattices. A (dual) **isomorphism of lattices** is a (dual) isomorphism of posets that also respects meets and joins. Formally, if (P, ≤) and (Q, ≤) are lattices and f: P→Q, then f preserves meets if f(ΛS) = Λf(S) for all S ∈ P, and f preserves joins if f(VS) = Vf(S) for all S ∈ P.



Closure Operators in Order Theory

DEFINITION [4]: Let $(P, \leq)$ be a poset. If $f: P \to P$ is an isotone, extensive, and idempotent function, then f is called a **closure map** on $(P, \leq)$.

REMARK: Closure maps can be interpreted as equivalence relations.

REMARK: Closure maps are tied to the topology of a poset.

EXAMPLE [4]: The function $g: (\mathbb{R} \cup \{-\infty,\infty\}) \to (\mathbb{R} \cup \{-\infty,\infty\})$ such that $g: x \mapsto \lceil x \rceil$ is a closure map on $(\mathbb{R}, \leq)$.

Galois Connections Define Closure Maps

DEFINITION [5]: (*Recall*) A **Galois Connection** between a poset $(P, \leq)$ and a poset $(Q, \leq)$ is a pair of of antitone functions $f: P \to Q$ and $g: Q \to P$ such that $x \leq (g \circ f)(x)$ for all $x \in P$ and $y \leq (f \circ g)(y)$ for all $y \in Q$.

LEMMA [1]: If $\langle f,g \rangle$ forms a Galois Connection between a poset $(P, \leq)$ and a poset $(Q, \leq)$, then $(g \circ f)$ is a closure map on $(P, \leq)$ and $(f \circ g)$ is a closure relation on $(Q, \leq)$.

THEOREM [1]: If $\langle f,g \rangle$ forms a Galois Connection between a complete lattices $(P, \leq)$ and $(Q, \leq)$, then $((g \circ f)(P), \leq)$ and $((f \circ g)(Q), \leq)$ are complete lattices with dual isomorphisms f and g.

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A Toy Galois Connection Between ℝ and ℤ

Proving Theorem 1 in beyond the scope of this presentation. Instead an example of Theorem 1 is presented.

EXAMPLE [5]. *The functions* h: $\mathbb{R}\to\mathbb{Z}$ *and* k: $\mathbb{Z}\to\mathbb{R}$ *such that* h: $x \mapsto -\lceil x \rceil$ *and* k: $x \mapsto -x$ *form a Galois Connection between the complete lattices* $(\mathbb{R} \cup \{-\infty,\infty\}, \leq)$ *and* $(\mathbb{Z} \cup \{-\infty,\infty\}, \leq)$. *The closure operators are* $(k \circ h)(x) = \lceil x \rceil$ *and* $(h \circ k)(x) = x$. *The induced lattices are both isomorphic to* $(\mathbb{Z} \cup \{-\infty,\infty\}, \leq)$.

$((\mathbb{R} \cup \{-\infty,\infty\})(k \circ h), \leq)$:

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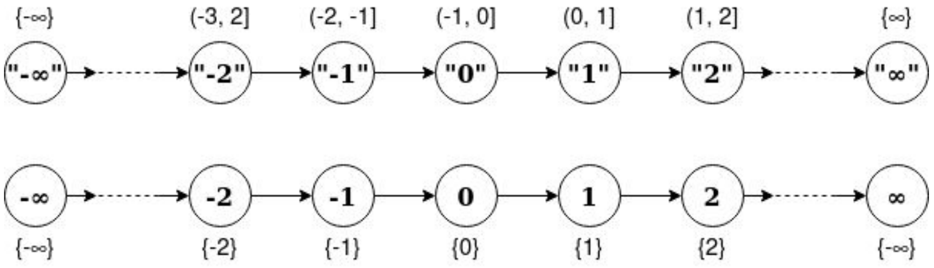

REMARK: The lattices induced by Galois connections are analogous to quotients by equivalence relations.

QUESTION: How do we find Galois Connections on lattices?

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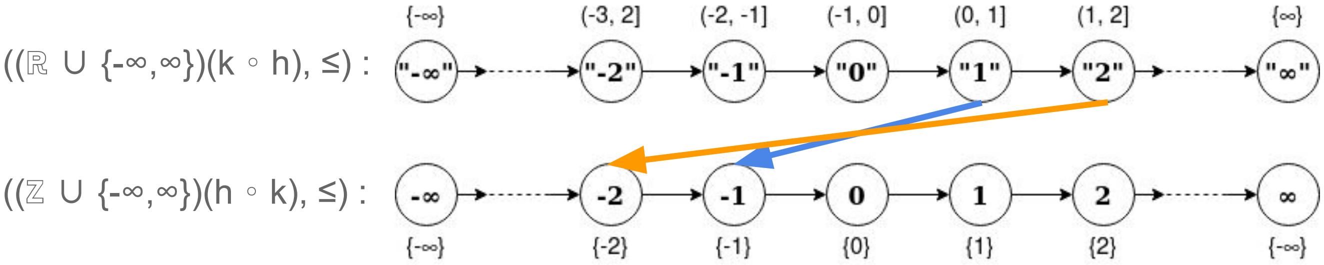

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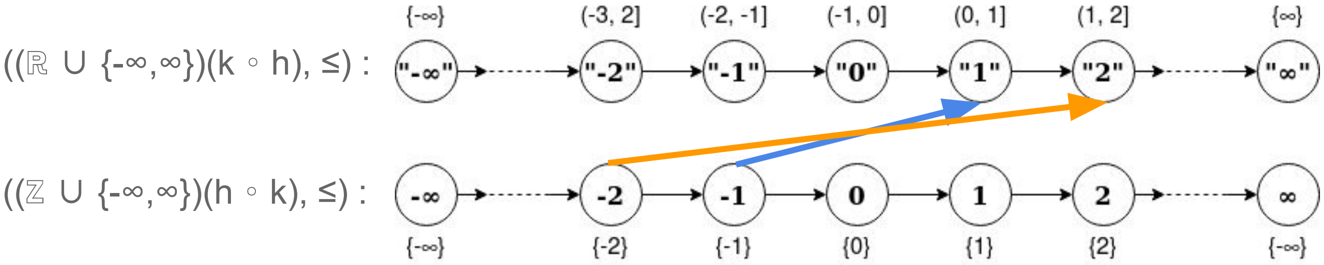

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Relations on Sets and Their Lattices

Relations on Sets Extend to Functions on Sets

DEFINITION [6]: Let X and Y be sets. A **relation** between X and Y is a subset R of X×Y. The **opposite relation** to R is a subset R^{op} of Y×X such that $(x, y) \in R$ if and only if $(y, x) \in R^{op}$.

REMARK: Every relation R between X and Y induces a unique function $f: \mathcal{P}(X) \to \mathcal{P}(Y)$. In particular, $f(\varnothing) = X$.

EXAMPLE [6]: *Consider the relation* R *between* $\mathbb{N}$ *and* $\mathbb{N}$ *such that* xRy *if and only* if $x \mid y$.

1. *If* $f: \mathcal{P}(X) \to \mathcal{P}(Y)$ *is induced by* R *and* $A \in \mathcal{P}(X)$, *then* f(A) *is the set of all common multiples of* A.
2. *If* $g: \mathcal{P}(Y) \to \mathcal{P}(X)$ *is induced by* R^{op} *and* $B \in \mathcal{P}(Y)$, *then* g(B) *is the set of all common divisors of* B.
3. *Furthermore*, $(g \circ f)(A)$ *contains the elements of* A *together with the divisors common to all elements of* A.

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Relations Induce Closures and Connections

THEOREM [2]: Let R be a relation between X and Y. If f: $\mathcal{P}(X)\to\mathcal{P}(Y)$ is the function induced by R and g: $\mathcal{P}(Y)\to\mathcal{P}(X)$ is the function induced by R^{op}, then $\langle f,g\rangle$ forms a Galois Connection between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$.

THEOREM [3]: Let R be a relation between X and Y. If f: $\mathcal{P}(X)\to\mathcal{P}(Y)$ is the function induced by R and g: $\mathcal{P}(Y)\to\mathcal{P}(X)$ is the function induced by R^{op}, then $(g \circ f)$ is a closure map for the lattice $(\mathcal{P}(X), \subseteq)$ and $(f \circ g)$ is a closure map for the lattice $(\mathcal{P}(X), \subseteq)$.

Closure Maps	Galois Connections
An isotone, extensive, and idempotent mapping of posets.	A pair of of antitone maps f: $P\to Q$ and g: $Q\to P$ such that $id_P \leq g \circ f$ and $id_Q \leq f \circ g$.

EXAMPLE [7]. The functions f and g from **EXAMPLE [6]** form a Galois Connection.

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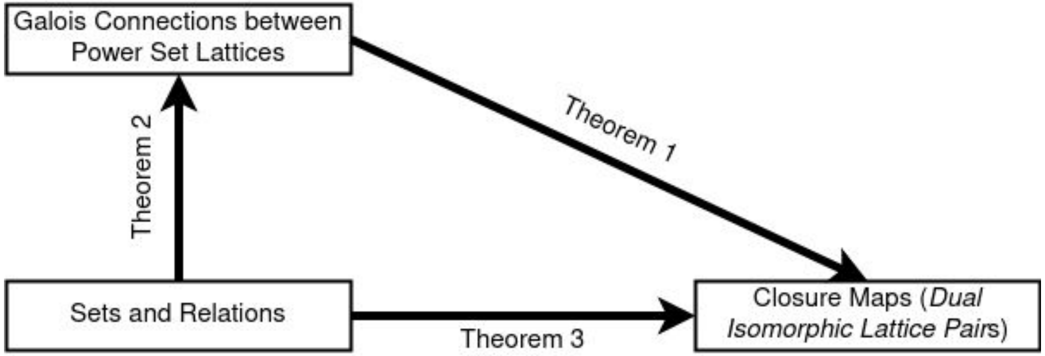
Do These Constructions Commute?



COROLLARY [1]: Let R be a relation between X and Y. If ⟨f,g⟩ is the Galois Connection induced by R, then the closure map induced by R is the closure map induced by ⟨f,g⟩.

An Order-Theoretic View of Logic

Boolean Algebras Are Lattices

Boolean algebras are equivalent to **complemented distributive** lattices. A lattice is distributive if meet and join satisfy the following distributive laws:

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z) \qquad X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$$

A lattice is complemented if every element x has a **complement** x' such that $x \wedge x' = 0$ and $x \vee x' = 1$.

One can prove that complemented distributive lattices satisfy the following properties (see Roman, pp. 127).

1. $0' = 1$, $1' = 0$, and $a'' = a$.
2. $(a \wedge b)' = a' \vee b'$, $(a \vee b)' = a' \wedge b'$.
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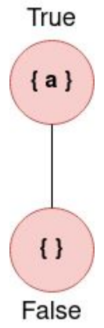
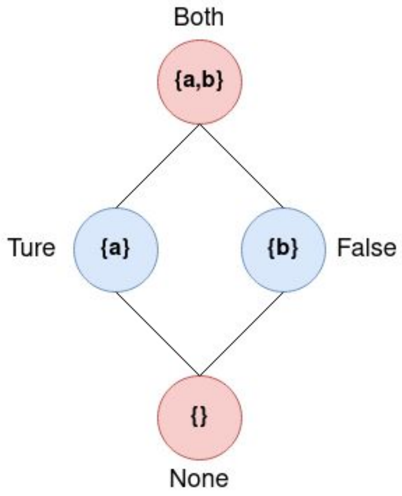
1. $\neg\bot = \top$, $\neg\top = \bot$, and $\neg(\neg a) = a$.
2. $\neg(a \wedge b) = \neg a \vee \neg b$, $\neg(a \vee b) = \neg a \wedge \neg b$.
3. $a \Rightarrow b$ if and only if $\neg a \vee b = \top$.

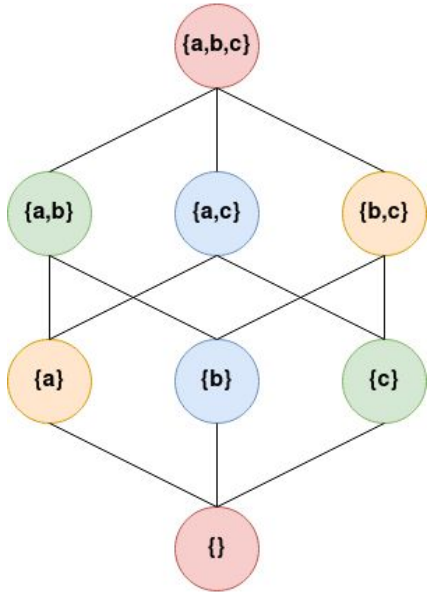
Power Set Lattices As 2^N-Valued Logic



Classical Logic



Reasoning with Contradictory Data



Reasoning under Uncertainty

Quantum Logic and Lattices

In quantum logic we consider a complete lattice (L, ≤) of **observable events** and a set $S \subseteq [0, 1]^L$ of (typically pure) **states**. Then ($\mathcal{P}$(S), ⊆) is a lattice of mixed states and ($\mathcal{P}$(L), ⊆) is a lattice of conjunctions of events.

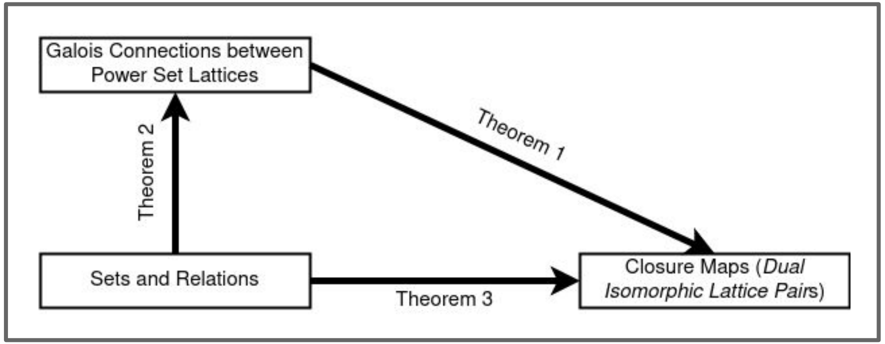
The relation R from S to L such that σRa if and only if σ(a) = 1 is used to determine observations that are made certain by σ. The closure map induced by R is related to **superposition**, and is be used to study **inaccessibility**.

At the time that [*Butterfield and Melia 1993*] was written, there was question as to the utility of **Birkhoff-von Neumann quantum logic**. This paper looked to decompose quantum logic into a hierarchy of assumptions, and to determined (through Galois Connections) what could be deduced under each collection of assumptions.

Birkhoff-von Neumann quantum logic has since been abandoned for a new account logical account of quantum mechanics based upon ***-autonomous categories** (i.e., linear logic).

Conclusion and Summary

- Galois correspondences generalize to order-theory and induce topological structures on the lattices.
- In the special case of power set lattices, relations give rise to both Galois Connections and closures
- Logic and logical relations can be understood as lattices with special structure
- Problems in quantum logic can be reframed as constructing Galois Connections between lattices.



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2. $\neg(a \wedge b) = \neg a \vee \neg b$, $\neg(a \vee b) = \neg a \wedge \neg b$.
3. $a \Rightarrow b$ if and only if $\neg a \vee b$.

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