Math 3032 Lecture 10 (23 Feb 2021)

Scheduling announcements:

- This week: shortened lectures to make time for OH type questions.
- HW 5 due on Thursday.
- Week of March 8 (week after next) no classes. optional review HW set that week

Today: Ideals. ← the most important notion in ring thy.

Motivation for ideals: Recall:

First isomorphism thm / homomorphism thm (for gps)

Suppose $\varphi : G \to H$ is a homomorphism of gps.
(in particular, G, H are gps. I'll write them multiplicatively)

Then $\ker(\varphi) := \varphi^{-1}(1_H) \subseteq G$

(a) is a normal subgp.

(b) and $G/\ker(\varphi) \cong \text{image}(\varphi)$,

(c) For every normal subgp N, there is a quotient gp G/N and a homomorphism $G \overset{\varphi}{\twoheadrightarrow} G/N$ with $\text{Kernel} = N$.

i.e. $\text{image}(\varphi) = G/N$

Slogan of the thm:

surjective homs $\equiv$ normal subgps.

This G/N is "unique"

To achieve an analogous thm for rings, we should ask: what structure has the kernel of a homorphism of rings?

both rings

Recall: A ring homorphism $\varphi : R \to S$ is a function s.t.

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad \text{additive}$$

$$\varphi(ab) = \varphi(a)\,\varphi(b) \qquad \text{multiplicative}$$

(sometimes we also ask $\varphi(1_R) = 1_S$ unital)

Additivity $\iff$ φ is a homorphism of additive grps.

$$\ker(\varphi) := \varphi^{-1}(0) \subseteq R$$

$$= \{ r \in R \text{ s.t. } \varphi(r) = 0 \}.$$

Recall: A ring homomorphism $\varphi : R \to S$ is

a function s.t. $\varphi(a+b) = \varphi(a) + \varphi(b)$ additive

$\varphi(ab) = \varphi(a)\,\varphi(b)$ multiplicative

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Additivity $\iff$ φ is a homomorphism of additive gps.

$$\ker(\varphi) := \varphi^{-1}(0) \subseteq R$$
$$= \{ r \in R \text{ s.t. } \varphi(r) = 0 \}.$$

$\Rightarrow$ • $(\text{image}(\varphi), +) \cong (R,+)/\ker(\varphi)$ iso of additive gps.

• $(\ker(\varphi), +)$ is a normal subgp of $(R,+)$

Since $(R,+)$ abelian: normal subgp $\equiv$ subgp.

If $\varphi: R \to S$ is a ring hom,

additivity $\Longrightarrow$ $\ker(\varphi)$ is an additive subgp of R.

Interesting part is multiplicativity.

$$\varphi(\underbrace{ab}_{\substack{\uparrow \\ R}}) = \varphi(a) \cdot \varphi(b)$$

mult. in S.

$\ker(\varphi) \ni a$

$\Updownarrow$

$\varphi(a) = 0$ (not 1)

So $\ker(\varphi)$ is related to $\cdot$ in R

$\longleftrightarrow$ 0 is related for $\cdot$ in S.

$$0 \cdot s = 0 = s \cdot 0 \quad \text{for any } s \in S.$$

$\Longrightarrow$ if $a \in \ker(\varphi)$ then $ab \in \ker(\varphi)$ $\forall b \in R$

i.e. if $\varphi(a) = 0$ because $\varphi(ab) = \varphi(a)\varphi(b) = 0 \cdot \varphi(b)$

Defn: A (two-sided) *ideal* in R is an additive subgp $I \subseteq R$ s.t.

$$\forall\, b \in R \text{ and } \forall\, a \in I, \quad ab,\ ba \in I.$$

i.e.

$\forall\, b \in R$, $bI := \{ b \cdot a \text{ for } a \in I \}$ and $Ib := \{ a \cdot b \text{ for } a \in I \}$ are subsets of I.

On previous slide: if $\varphi : R \to S$ is a homomorphism, then $\ker(\varphi)$ is an ideal.

Remark: "ideal" $\equiv$ two-sided ideal.

there are also one-sided ideals.

e.g. a *left ideal* is $I \subseteq R$ additive subgp s.t. $aI \subseteq I$ $\forall\, a \in R$.

Justify the definition: Want to show that every ideal is a kernel of a ring hom.

Strategy: Given $I \subseteq R$ an ideal, already have an additive gp R/I.

(because I is a subgp of abelian gp $(R,+)$, and subgps of abelians are normal)

We will give R/I a ring structure s.t. the quotient map
$$R \to R/I$$
$$r \mapsto [r] := \underbrace{r+I}_{\text{coset}}$$
is a ring map.

This R/I will be the <u>quotient ring</u> aka <u>factor ring</u>.

Elements of $(R/I, +)$ are cosets $\underset{[r]}{\overset{r+I}{\shortparallel}} \subseteq R$.

where $[r] = [r']$ iff $r' - r \in I$.

If $R \to R/I$ is going to be a ring map,
$r \longmapsto [r]$

then mult in R/I will have to be

$$[r] \cdot [s] := [r \cdot s]$$

i.e. $$(r+I)\cdot(s+I) := rs + I.$$

This will work provided that if $[r] = [r']$ and $[s] = [s']$ then $[r \cdot s] = [r' \cdot s']$

This will work provided that if $[r]=[r']$ and $[s]=[s']$ then $[r\cdot s]=[r'\cdot s']$

WTS: if $r'-r\in I$ and $s'-s\in I$ then $r'\cdot s'-r\cdot s\in I$.

Pf: Rewrite

$$r'\cdot s' - r\cdot s = r'\cdot s' \overbrace{- r's + r's}^{0} - rs.$$

$$= r'\cdot\underbrace{(s'-s)}_{I} + \underbrace{(r'-r)}_{I}s$$

Since $s'-s\in I$, $r'\cdot(s'-s)\in I$

Since $r'-r\in I$, $(r'-r)\cdot s\in I$

Since $(I,+)$ is a gp, RHS $\in I$. $\square$.