Math 3032 (16 March 2021)

OH: Today 12–2pm

HW 7: Due Thursday.

Optional HW: due Tuesday.

finitely many variables.

$\mathbb{F}$ is a field.

Goal: "Simplify" ideals in $\mathbb{F}[\overbrace{x, y, \dots}]$

$\hookrightarrow$ not a precise term. If you find yourself faced with an ideal, how to describe it succinctly?

Example: If only one variable, then $\mathbb{F}[x]$ is a PID. Given $I \subseteq \mathbb{F}[x]$, can use long div to find $p(x) \in \mathbb{F}[x]$ s.t. $I = \langle p \rangle$.

Why?. Ideals $\leftrightsquigarrow$ systems of polynomial equations.
in $\mathbb{F}[x, \dots]$

$$\langle f(x,y), g(x,y) \rangle \longleftarrow \left[\begin{array}{l} f(x,y) = 0 \\ g(x,y) = 0 \end{array} \right.$$

for any $h(x,y) \in$, if (x,y) solves , then it solves eqn $h(x,y) = 0$ as well.

If we can find a simpler description of $\langle f, g \rangle$, e.g. maybe $\langle f', g' \rangle$ where f' and g' are "simple" than f, g. then we will have simplified our description of the solution set.

Elementary example:

(linear)

in $\mathbb{R}[x,y,z]$

$$f(x,y,z) := 3x - y + z - 1$$
$$g(x,y,z) := x + 2y + z - 6$$

replace $f \rightsquigarrow f - 3g$.

$$g = x + 2y + z - 6$$
$$f - 3g = -7y - 2z + 17 =: h$$

$$\langle f, g \rangle = \langle g, f - 3g \rangle.$$

$$h \rightsquigarrow \frac{h}{-7} = y + \frac{2}{7} z + \frac{17}{7}$$

$$\langle g, h \rangle = \langle g, h/-7 \rangle.$$

$$g \rightsquigarrow g - 2 \cdot \frac{h}{-7} = x + 0y + \# z + \#.$$

the system $f(x,y) = g(x,y) = 0$ of inhomogeneous linear equations

Solving these is the topic of Linear Algebra.

$$\begin{pmatrix} 3 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

method from linear alg:

row-echelon reduction.

In terms of poly equation this is: ...

Note: we arbitrarily chose order x, y, z.

$\vec{x} := (x_1, \dots, x_n)$ "$f_i(\vec{x})$"

Defn (name is not great):

A **basis** for an ideal $I \subseteq \mathbb{F}[x_1, x_2 \dots x_n] = R$ is a set $f_1, f_2, \dots, f_r \in R$ s.t. $I = \langle f_1, \dots, f_r \rangle$.

not typically equal

$$= \{ c_1(\vec{x}) f_1(\vec{x}) + c_2(\vec{x}) f_2(\vec{x}) + \dots + c_r(\vec{x}) f_r(\vec{x})$$

where $c_i(\vec{x})$ vary over $\mathbb{F}[\vec{x}]$. $\}$.

N.B.: We are not assuming any "linear independence". Better name would be "generating set".

If c_i's were scalars rather than polynomials, then this would look like the span of a basis.

Hilbert basis theorem: Every ideal in $\mathbb{F}[x_1, \ldots, x_n]$ does admit a finite basis.

Improved goal: given a basis, find a "simpler" basis for the same ideal.

The following operations take bases to bases (for same I):

- reordering.
- removing 0.
- rescale basis elts by nonzero $\overbrace{\text{numbers}}^{\text{elts of } \mathbb{F}}$.
- replace $\overbrace{f_1(\vec{x})}^{=f}$ by $f_1(\vec{x}) + c(\vec{x})\overbrace{f_2(\vec{x})}^{=g}$.

e.g. in $\mathbb{F}[x]$, $f \mapsto$ remainder in long div of f by g.

- add a new elt from I to the basis.

Method: Do this "replacement"

$$f_1(\vec{x}) \longmapsto f_1(\vec{x}) + c(\vec{x})\, f_2(\vec{x}) =: f_1'(\vec{x})$$

so that $f_1'(\vec{x})$ is "smaller" than $f_1(\vec{x})$.

E.g.: in one variable, "smaller" meant "lower degree".

Linear case: "smaller" meant e.g. "lower degree in x".

"order of magnitude"

Imagine $\boxed{x \gg y \gg z} \gg 1$ ordering is arbitrary.

nonzero constants ≈ 1

$7x \approx x$

$x + 2y \approx x$

$x + 2y + z + 1 \gg 2y + z + 1$

$\gg z + 1$

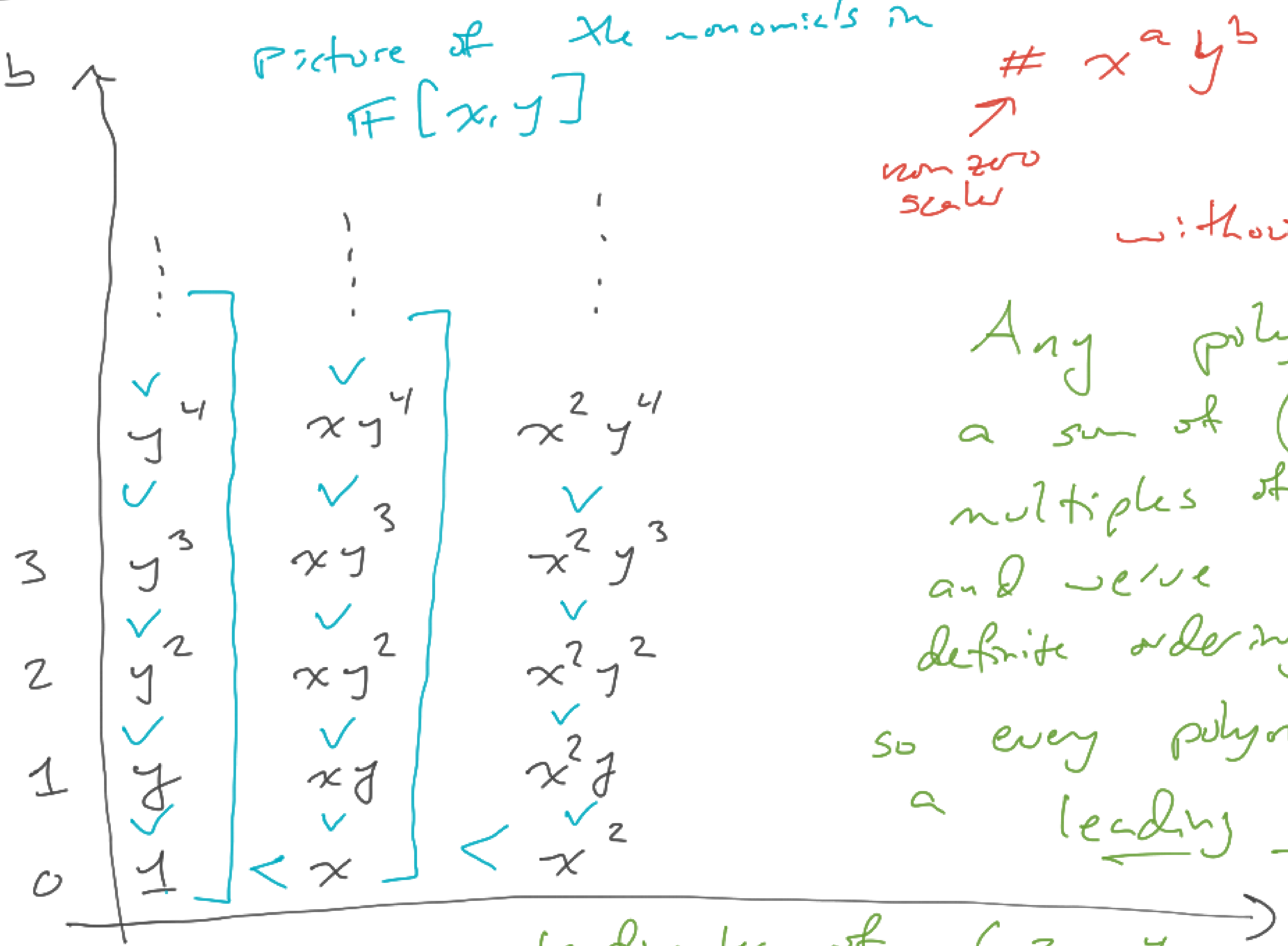
$\gg 1.$

N.B: $x \gg y$

"so much bigger" that $y^2 \gg y$.

also $x \gg y^2$ and $x \gg y^3$ and so on.

Picture (two variable case):



A *monomial* is

$\# \, x^a y^b \qquad a,b \in \mathbb{N} = \{0,1,2,\dots\}$

nonzero scalar

without sums.

Any polynomial is a sum of (nonzero scalar multiples of) monomials, and we've decided a definite ordering of monomials, so every polynomial has a leading term.

e.g. leading term of $(y^2+y^4+xy^2+x^2)$ is x^2

- Any poly is $\approx$ its leading term.

$$f(x,y,\dots) = \underbrace{\text{leading monomial}}_{\approx f} + \underbrace{\text{lower order}}_{\ll f}$$

- leading term of $(f(x,y)\cdot g(x,y))$

$= (\text{leading term of } f)\cdot(\text{leading term of } g)$.

Our simplification method will try to replace basis elts by elts with smaller leading term.

Best situation:

$$I = \langle \underbrace{\overset{\approx xy^2}{xy^2+y^3+3}}_{g(x,y)},\ \underbrace{\overset{\approx x^2y^3 = (xy)(xy^2)}{x^2y^3+x^2+2}}_{f(x,y)} \rangle.$$

Because leading terms divide cleanly, long division is very attractive. $\leftarrow$ or rather, 1st step.

$$f(x,y) \rightsquigarrow f(x,y) - xy\cdot g(x,y).$$

Best situation: $I = \langle xy^2 + y^3 + 3,\ x^2y^3 + x^2 + 2 \rangle$

$ \underset{g(x,y)}{\shortparallel} \qquad \underset{f(x,y).}{\shortparallel}$

Because leading terms divide cleanly, long division is very attractive. ← or rather, 1st step.

$$f(x,y) \rightsquigarrow f(x,y) - xy \cdot g(x,y).$$

$$= \underbrace{x^2 - xy^4 - 3xy + 2}_{h(x,y)}$$

$$I = \langle x^2 - xy^4 - 3xy + 2,\ xy^2 + y^3 + 3 \rangle.$$

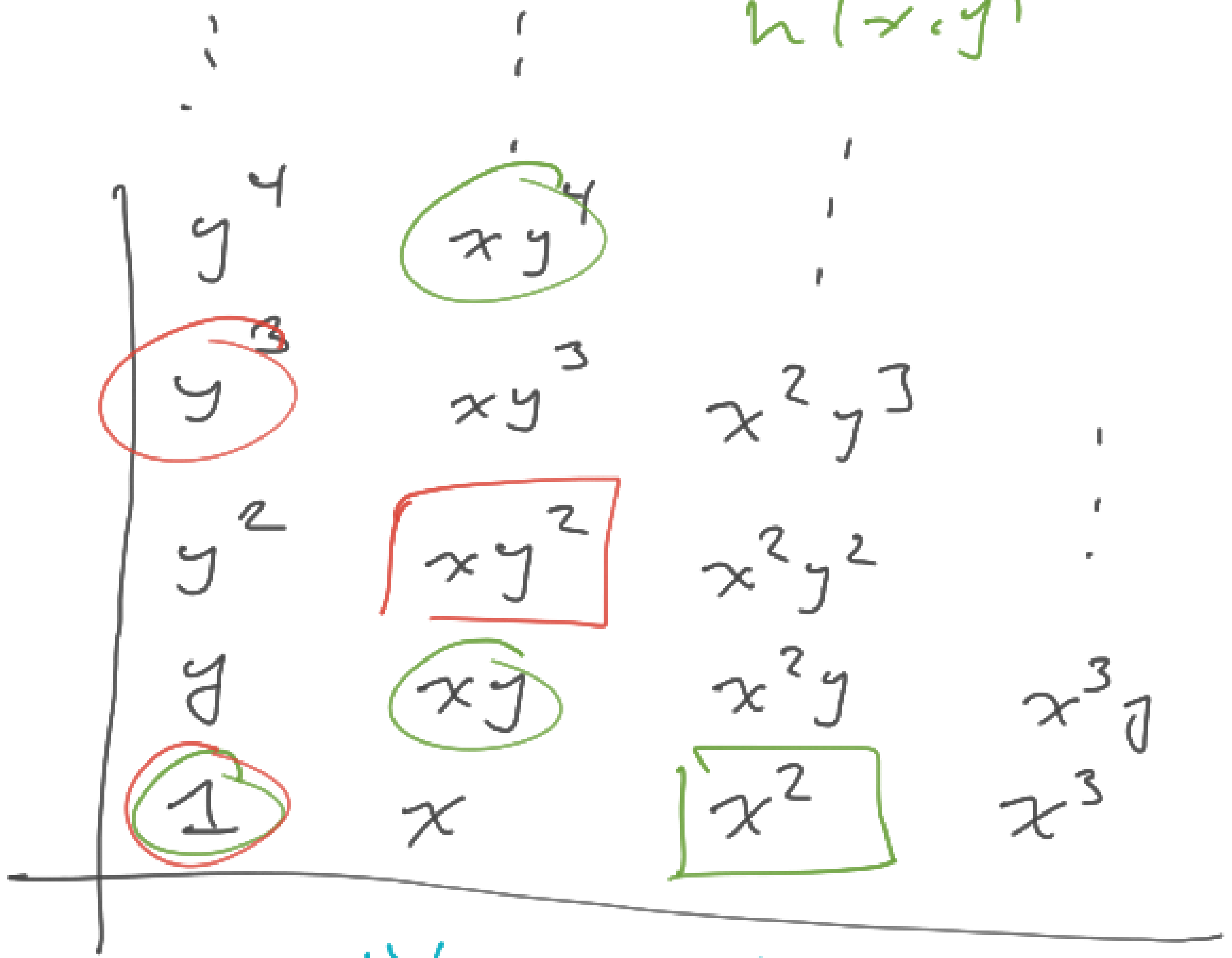
Now we have to decide how to proceed.

$h >> g$, so $g \rightsquigarrow g + c \cdot h$ will increase g. actually bad.

$h \rightsquigarrow h + cg$ will be of the same size. doesn't help.

$$I = \langle x^2 - xy^4 - 3xy + 2,\ xy^2 + y^3 + 3 \rangle \subseteq \mathbb{Q}[x,y]$$

$\underbrace{x^2 - xy^4 - 3xy + 2}_{= h(x,y)}$, $\underbrace{xy^2 + y^3 + 3}_{= g(x,y)}$



"l.c.m of h, g"

$s(x,y) = x \cdot g(x,y) - y^2 \cdot h(x,y)$.

leading term of h bigger than, but not divisible by leading term of g.

Vocab aside:

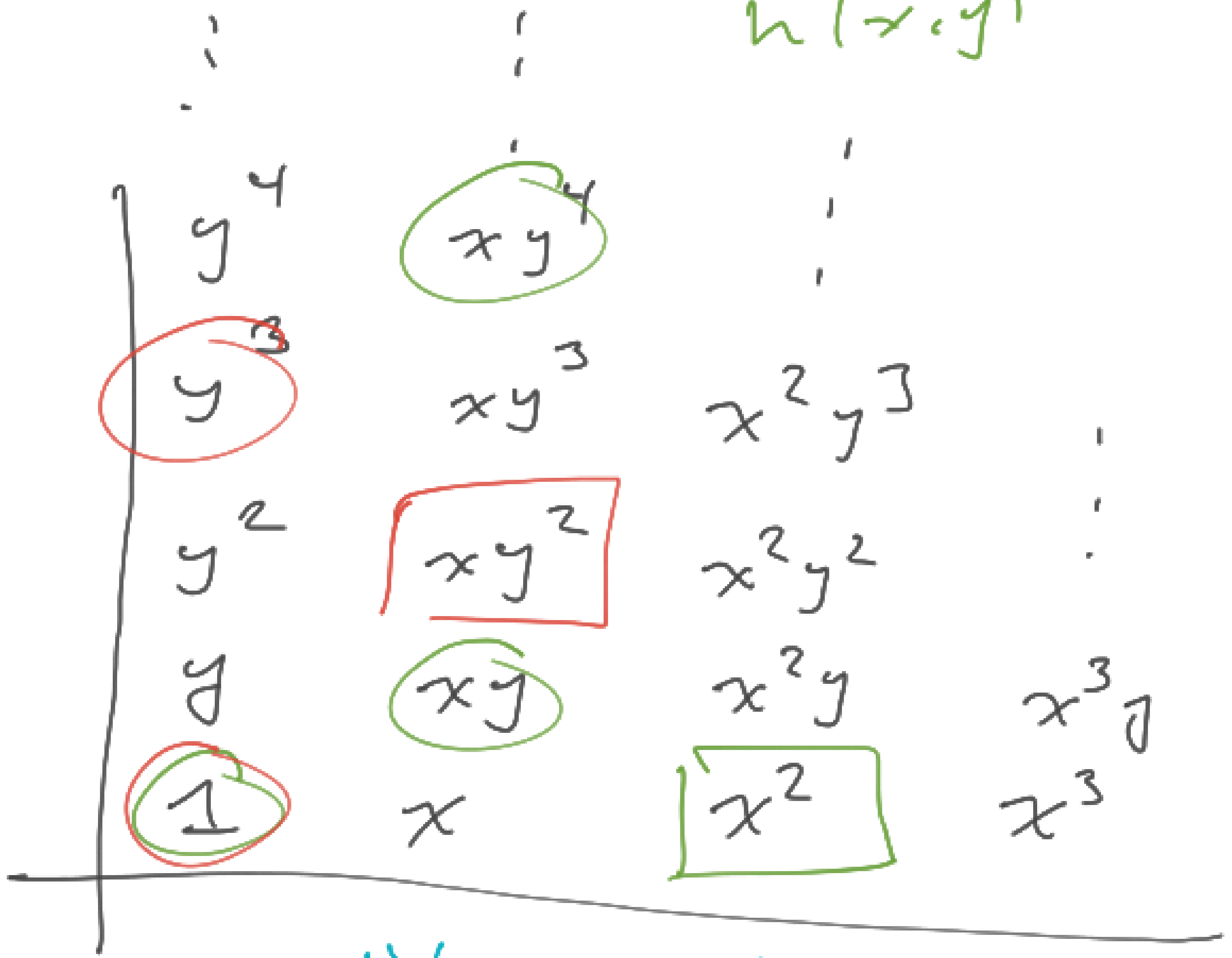
A total order on a set is where any two elts are comparable: either $a<b$ or $a>b$ or $a=b$.

if $a>b$ and $b>c$ then $a>c$.

e.g. set of monomials

divis. A partial order allows $a \neq b$ and $a \not< b$ and $b \not< a$.

$$I = \langle \underbrace{x^2 - xy^4 - 3xy + 2}_{=\,h(x,y)},\ \underbrace{xy^2 + y^3 + 3}_{=\,g(x,y)} \rangle \subseteq \mathbb{Q}[x,y]$$



"l.c.m of h, g"

$$s(x,y) = x\cdot g(x,y) - y^2\cdot h(x,y)$$

$$s(x,y) = x^2y^2 + xy^3 + 3x - (x^2y^2 - xy^6 - 3xy^2 + 2y^2)$$

$$= xy^6 + xy^3 + 3xy^2 + 3x + 2y^2$$

$$I = \langle h(x,y),\ s(x,y),\ g(x,y) \rangle$$

$$s \rightsquigarrow s - y^4 \cdot g$$

$$I = \langle \overset{\substack{h(x,y) \\ \shortparallel}}{x^2 - xy^4 - 3xy + 2},\ \overset{\substack{g(x,y) \\ \shortparallel}}{xy^2 + y^3 + 3} \rangle \subseteq \mathbb{Q}[x,y]$$

$$s(x,y) = xy^6 + xy^3 + 3xy^2 + 3x + 2y^2$$

$$I = \langle h(x,y),\ s(x,y),\ g(x,y) \rangle$$

$$s \rightsquigarrow \underset{\substack{\shortparallel \\ t(x,y)}}{s - y^4 g} \qquad y^4 g = xy^6 + y^7 + 3y^4$$

$$t(x,y) = xy^3 + 3xy^2 + 3x - y^7 - 3y^4 - 2y^2$$

$$t \rightsquigarrow t - yg \qquad - xy^3 - y^4 - 3y$$

$$u(x,y) = 3xy^2 + 3x + \dots \quad \text{l.o.}$$

$$u(x,y) \rightsquigarrow u - 3g = 3x + \text{l.o.}$$