3032 Lecture 4 (Jan 21)

Reminder: HW 1 due by end of day today.

OH 1-3pm today in this Collaborate classroom.

Starting next week, OH 2-4pm Tuesdays.

HW2 on brightspace, due next week.

Today: Fractions.

From here on out, "rings" will almost always be assumed unital, and usually commutative.

Suppose R is a (commutative, unital) ring and $s \in R$.

<u>Question</u>: Can we "expand" R to a new ring "$R[s^{-1}]$" in which s is invertible?

If so, what is the smallest such expansion? (if there is a "smallest" one)

<u>E.g.</u>: $\mathbb{Z}[10^{-1}] = \{x \in \mathbb{R}$ s.t. the decimal expansion of x terminates$\}$.

<u>Note</u>: Will never work if s is a zero divisor.

Indeed, suppose $\boxed{sb = 0 \quad b \neq 0}$ $\in$ "$R[s^{-1}]$" and s is invertible in $R[s^{-1}]$.

Then $b = s^{-1}sb = s^{-1}0 = 0$.

What elements are in $R[s^{-1}]$?

(0) Each $r \in R \rightsquigarrow$ copy in $R[s^{-1}]$. "$\frac{r}{1}$" $s^0 = 1$.

(1) Each $r \in R \rightsquigarrow$ "$\frac{r}{s}$" $\in R[s^{-1}]$.

(2) "$\frac{r}{s^2}$", etc.

In other words, for each pair (r, s^k) $r \in R$, $k \in \mathbb{N}$

there should be an element "$\frac{r}{s^k}$".

$\underline{\underline{\mathbb{N} = \{0, 1, 2, 3, \ldots\}}}$

But there's redundancy! For example, should have

$$\frac{s}{s} = 1 = \frac{1}{1} \text{ in } R[s^{-1}]. \qquad \frac{rs}{s} = \frac{r}{1} \;, \ldots$$

Dealing with redundancy is the raison d'etre for equivalence relations.

$\{1, s, s^2, s^3, \ldots\} \overset{?}{\cong} \mathbb{N}$

$1 = s^0$

So, let's take the set of pairs $\{(r, s^k) : r \in R, k \in \mathbb{N}\}$

and define an equivalence relation

idea: "supposed to be" $\frac{r}{s^k} = \frac{r'}{s^{k'}}$.

$(r, s^k) \sim (r', s^{k'})$ if $\frac{r}{s^k} = \frac{r'}{s^{k'}}$. i.e.

if $r s^{k'} = s^k r'$.

Defn: $R[s^{-1}] = \{\text{equiv classes}\}$.

Is $\sim$ an equivalence relation?

- $(r, s^k) \sim (r, s^k)$? Yes: because $rs^k = rs^k$.

- sym? Yes ✓

- transitive?

$(r, s) \sim (r', s')$ if $rs' = r's$.

$$(r, s^k) \sim (r', s^{k'}) \sim (r'', s^{k''})$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$s^{k''} r s^{k'} = s^{k''} r' s^k = r' s^{k''} s^k = r'' s^{k'} s^k$$

Because s is not a zero divisor,

$$\Rightarrow \quad rs^{k''} = r'' s^{k'}. \quad ✓$$

The symbol "$\frac{r}{s^k}$" is defined as the equivalence class of (r, s^k) under this equiv rel.

Is $R[s^{-1}]$ a ring?

- $\frac{r}{s^k} + \frac{r'}{s^{k'}} = \frac{rs^{k'} + r's^k}{s^k s^{k'}}$

- $\frac{r}{s^k} \cdot \frac{r'}{s^{k'}} = \frac{rr'}{s^k s^{k'}} = \frac{rr'}{s^{k+k'}}$.

Make sure these are well-defined!

Check: if $(r, s^k) \sim (r'', s^{k''})$, is $(rr', s^{k+k'}) \overset{?}{\sim} (r''r', s^{k''+k'})$?

$\underbrace{rs^{k''} = r''s^k}$

$\times\ r's^{k'}$

$\Downarrow$

$rr's^{k''+k'} \overset{?}{=} r''r's^{k''+k'}$ ✓

Checking the ring axioms is not very enlightening.

assoc. of $+$? $\quad \frac{r}{s^k} + \frac{r'}{s^{k'}} + \frac{r''}{s^{k''}} = \frac{rs^{k'}s^{k''} + r's^k s^{k''} + r''s^k s^{k'}}{s^k s^{k'} s^{k''}}$

Proposition:

$R \hookrightarrow R[s^{-1}]$ ✓.

$R \notin R[s^{-1}]$ but close enough

(1) $R \to R[s^{-1}]$, $r \mapsto \frac{r}{1}$ is a ring homomorphism.

$$\frac{r}{1} + \frac{r'}{1} = \frac{r1 + r'1}{1 \cdot 1} = \frac{r+r'}{1} \checkmark.$$

(2) If s is not a zero-divisor, then $r \mapsto \frac{r}{1}$ is injective.

Recall: Any ring hom is a hom of additive gps.

A gp hom $f : (R, +) \to (R[s^{-1}], +)$ is injective iff its kernel $\mathrm{Ker}(f) = f^{-1}(0) = \{r \in R \text{ s.t. } f(r) = 0\}$ is trivial (i.e. $= \{0\}$).

So it suffices to ask which $r \overset{?}{\mapsto} \frac{0}{1} \in R[s^{-1}]$.

i.e. for which r is $(r, 1) \overset{?}{\sim} (0, 1)$. $\quad r \cdot 1 \overset{?}{=} 0 \cdot 1 \quad r = 0.$

$\square$

The construction $R \rightsquigarrow R[s^{-1}]$ works for any set $S \subseteq R$ of elements, to build "$R[S^{-1}]$".

contains no zero-divs.

$R[S^{-1}] \supseteq R$ should be a ring s.t. $\forall s \in S$, s is invertible in $R[S^{-1}]$.

<u>N.B</u>: If s, s' both invertible, so is ss'. So we might as well assume S is closed under $\times$, and $1 \in S$.

E.g: $R[s^{-1}] = R[S^{-1}]$ for $S = \{1, s, s^2, s^3, \ldots\} = s^{\mathbb{N}}$.

<u>Defn</u>: Suppose R is a (com. unital) ring and $1 \in S \subseteq R$ is closed under $\times$, and contains no zero divs.

Then $R[S^{-1}]$ is the set of "fractions" $\frac{r}{s}$ for $r \in R$, $s \in S$, more precisely the set of equivalence classes of ordered pairs (r,s) where

$$(r,s) \sim (r',s') \text{ if } rs' = r's.$$

Proposition:

(1) $R[S^{-1}]$ is a ring.

(2) $r \mapsto \frac{r}{1}$ is a ring hom. $R \to R[S^{-1}]$.

(3) $r \mapsto \frac{r}{1}$ is injective since S contains no zero divisors.

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \quad \text{etc.}$$

In particular, if R is a integral domain, then we can take $S = R \setminus \{0\}$.

$\uparrow$ zero is the only zero-div.

Proposition: In this case, $R[S^{-1}]$ is a field.

Pf: We know it's a ring. We just need to check that if $\frac{r}{s} \in R[S^{-1}]$, $\frac{r}{s} \neq 0$, then it's inv under $\times$. $\left(\frac{r}{s}\right)^{-1} = \frac{s}{r} \in R[S^{-1}]$ because $r \neq 0$.

Cor: Every domain embeds in a field.

In fact, R is a domain iff $\exists$ field $\mathbb{F}$ and embedding $R \hookrightarrow \mathbb{F}$.

= inj. hom.

Defn: $R[(R \setminus \{0\})^{-1}]$ is the field of fractions of R.

Example: Field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

Example: If R is a field.
Then field of fractions of R "is" R.

N.B.: Not equal as sets.

$r \mapsto \frac{r}{1}$ $\quad R \longrightarrow$ its field of fractions

is an ISO.

Inj. hom for any domain.

Obviously surjective: $rs^{-1} \mapsto \frac{r}{s}$ $\qquad rs^{-1}s \overset{?}{=} r \cdot 1$ ✓.

$$R \ni rs^{-1} \mapsto \frac{rs^{-1}}{1}$$

Universality: Let R and S as above.

com. unital ring. $S \subseteq R$ containing no zero-divs.

Suppose R' is a ring containing R s.t.

i.e. we're given an inj. hom $f: R \hookrightarrow R'$

every $s \in S$ is invertible in R'. Then $\underbrace{R[S^{-1}] \hookrightarrow R'}_{\text{unique}}$.

$f(s)$ is inv.

Pf: (1) Extend f to a hom. $\tilde{f}: R[S^{-1}] \longrightarrow R'$.

$$\tilde{f}\left(\frac{r}{s}\right) = f(r) \cdot f(s)^{-1} \in R'.$$

Well defined? If $\frac{r}{s} = \frac{r'}{s'}$.

i.e. $(r, s) \sim (r', s')$.

Then $f(r) f(s)^{-1} \overset{?}{=} f(r') f(s')^{-1}$

$f(r) f$

is $\tilde{f}$

Well defined? Need:

if $rs' = r's$, is $\overbrace{f(r)f(s)^{-1} \overset{?}{=} f(r')f(s')^{-1}}^{\in R'}$

mult both sides by $f(s)\,f(s')$ $\Updownarrow$

$$f(r)\,f(s') \overset{?}{=} f(r')f(s)$$

$$\begin{array}{ccc} f(r)f(s') & \overset{?}{=} & f(r')f(s) \\ \parallel & & \parallel \\ f(rs') & & f(r's). \end{array}$$

✓.

is $\hat{f}$ a ring hom?

yes.

(2) Is $\tilde{f}: R[S^{-1}] \to R'$ an injection? i.e. what is its kernel?

if $f(r)f(s)^{-1} = 0$, $\Rightarrow$ $f(r) = 0$. $\Rightarrow$ $r = 0$

$\frac{r}{s} \in \mathrm{Ker}(\tilde{f})$ $\Uparrow$ because f is inj. $\square$

In particular, if R is a domain, any field extension of R contains the field of fractions of R.

i.e. any

$$\begin{tikzcd} R \arrow[rr, hook] \arrow[dr, hook] & & \mathbb{F} \\ & \text{field of fractions} \arrow[ur, hook] & \end{tikzcd}$$

This characterizes the field of fractions up to unique isomorphism.