Math 3032 Lecture 5 (Jan 26)

Office hours today 2-4 pm here.

Goal: Given any ring R ⇝ produce a ring $R[x]$
ring of polynomials in one ~~variable~~ indeterminate
w/ coefs in R.

Idea is: Some typical elements in $R[x]$
include $1, x, 1+x, x^2, x^2-1, \ldots$

more generally, if $a_0, \ldots, a_n$ is a finite
sequence in R, then $\sum_{i=0}^{n} a_i x^i = \underbrace{a_0}_{1=x^0} + a_1 \overset{x^1}{x} + a_2 x^2 + \ldots + a_n x^n$
should be in $R[x]$.

Careful: $\{\text{finite sequences}\} \to R[x]$

$$(a_i) \longmapsto \sum_i a_i x^i$$

is not a bijection. It is a surjection, just not an injection.

$$(1, 1, 0) \longmapsto 1 + 1x + 0x^2$$

$$(1, 1) \longmapsto 1 + 1x$$

should be equal.

Some equivalent options:

(1) $\{\text{finite sequences in } R\}$ / equiv relation in which two finite sequences are equiv if the same up to appending 0s at the end.

i.e. declare: $\forall a_0, \dots, a_n$
$(a_0, \dots, a_n) \sim (a_0, \dots, a_n, 0)$

Finite sequences of the same length have +, and have to check compat with $\sim$.

(2) {infinite sequences which are $\underbrace{\text{eventually zero.}}$ } | i.e. $\{(a_i)_{i=0}^{\infty}$ s.t. $\exists n$ s.t. $a_i = 0 \ \forall i > n\}$.

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zero after some cut off

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zero for all but finitely many indices.

R[x] as an additive gp?

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i.$$

(2) Use entrywise addition on {infinite sequences}.

$\hookrightarrow$ check that if (a_i), (b_i) both eventually zero, then $(a_i + b_i)$ is eventually zero.

Aside: Suppose you have a infinite sequence of abelian gps, which I will write additively.

$A_0, A_1, A_2, \ldots$

In example, $A_i \cong (\mathbb{R}, +)$
$\shortparallel$
$\mathbb{R}\cdot x^i$ = set of monomials of deg. i.

Two "products" of the whole sequence.

(Compare: If we had finitely many A_i
e.g. A_0, A_1, A_2, A_3.
$\rightsquigarrow A_0 \times A_1 \times A_2 \times A_3 = \{$ordered tuples (a_0, a_1, a_2, a_3) s.t. $a_i \in A_i\}$.)

$A_2 \leftrightarrow$ $a_2 \mapsto (0, 0, a_2, 0)$

$\prod_{i=0}^{\infty} A_i = \{$sequences (a_i) s.t. $a_i \in A_i\}$.
$\cup\!\!\shortmid$
coproduct
$\coprod_{i=0}^{\infty} A_i = \{$sequences which are eventually zero$\} = \bigoplus_{i=0}^{\infty} A_i$.

$R[x] \underset{\text{as a gp}}{\cong} \coprod_{i=0}^{\infty} R x^i.$

polynomials $\sum_{i=0}^{n} a_i x^i$

$R[[x]] \cong \prod_{i=0}^{\infty} R x^i$

power series. (infinite sums allowed)

For any indexing set I, and any "sequence of ab. gps indexed by I" $i \mapsto A_i$

$\leftrightarrow$ a function $I \to \{\text{ab gps}\}$.

$$\coprod_{i \in I} A_i := \left\{ \begin{array}{l} \text{set of } I\text{-indexed sequences } i \mapsto a_i \\ \text{s.t. } a_i \in A_i \text{ for each } i \\ \text{and all but finitely many } a_i\text{'s are zero.} \end{array} \right\}$$

$R[x]$ as a gp:

$R = \mathbb{Z}$.

$(1+x) + (-x + x^2) = 1 + x^2$.

To make $R[x]$ into a ring,

we want to define multiplication: $x \cdot x = x^2$

$x^i \cdot x^j = x^{i+j}$ ...

$$\left(\sum_{i=0}^{m} a_i x^i\right) \cdot \left(\sum_{j=0}^{n} b_j x^j\right) := \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i b_j\right) x^k$$

$\cdot$	$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ ($\dots = 0$)
b_0	$a_0b_0 + a_1b_0x \quad\quad 0$
$+$	$+ \quad\quad +$
b_1x	$a_0b_1x + a_1b_1x^2 \quad\quad 0$
$+$	
b_2x^2	$a_0b_2x^2 + \quad 0$
$+$	
$0 = b_3x^3$	0
$+$	
$0 = \vdots$	
$0 =$	

(1) Because each diag is finite, $R[[x]]$ has a multiplication.

(2) Because the product of finite sets is finite, $R[x] \subseteq R[[x]]$ is a subring.

- Is x invertible in $R[x]$, $R[[x]]$?

No for both.

$$x \cdot (a_0 + a_1 x + \dots) \overset{?}{=} 1 = 1 \cdot x^0$$
$$\shortparallel$$
$$0 \cdot x^0 + a_0 x^1 + a_1 x^2 + \dots$$

cannot work because coef of x^0 are dif.

- Is $(x+1)$ invertible in $R[x]$, $R[[x]]$?

No for $R[x]$.

Yes for $R[[x]]$.

$$(1+x)(a_0 + a_1 x + a_2 x^2 + \dots) \overset{?}{=} 1$$
$$\shortparallel$$
$$\underset{\underset{1}{\shortparallel}}{a_0} + \underbrace{(a_0 + a_1)}_{\underset{0}{\shortparallel}} x + (a_1 + a_2) x^2 + \dots$$

$$0 \Rightarrow a_1 = -1 \qquad \Rightarrow a_2 = +1$$

$$1 - x + x^2 - \dots = \sum_{i=0}^{\infty} (-1)^i x^i = (1+x)^{-1} \in R[[x]].$$

Jia: An inverse for x could be

$$\sum_{i=0}^{\infty} (-1)^i (1-x)^i \overset{?}{\in} R[[x]] \quad \text{No.}$$

$$\shortparallel$$

$i=0:\ 1$

$i=1:\ x-1$

$i=2:\ x^2-2x+1$

$i=3:\ x^3-3x^2+3x-1$

$i=4:\ x^4-4x^3+6x^2-4x+1$

$\cdots$

$$= (1-1+1-1+\ldots)x^0 + (1-2+3-4+\ldots)x^1 + (1-3+6-\ldots)x^2$$

One more variation: "Divided powers":

Sometimes in calculus you want $\sum a_i x^i$.

often: $\sum \frac{c_n}{n!} x^n$ e.g. • Taylor's formula
• exp.

$\coprod_{i=0}^{\infty} R x^i$ $R[x]$ as a gp

has another interesting multiplication where you think of $(a_i) \rightsquigarrow \sum \frac{a_i}{i!} x^i$.

amusing exercise to show is assoc.

$$\left(\sum a_i \frac{x^i}{i!} \right) \left(\sum b_j \frac{x^j}{j!} \right) = \sum_k \left(\sum_{i+j=k} \binom{k}{i} a_i b_j \right) \frac{x^k}{k!}$$

Proposition: If R is a domain then so are $R[x]$ and $R[[x]]$.

Pf: • $R[x] \subseteq R[[x]]$ so it suffices to do the second one.

• Assume $\overset{a(x)}{\overset{\shortparallel}{\sum a_i x^i}}$ and $\overset{b(x)}{\overset{\shortparallel}{\sum b_j x^j}}$ both not zero. want to show that their product is not zero.

There is a first nonzero entry in (a_i), (b_j).

$a(x) = \underbrace{a_p}_{\neq 0} x^p + a_{p+1}x^{p+1} + \dots \qquad b(x) = \underbrace{b_q}_{\neq 0} x^q + b_{q+1}x^{q+1} + \dots$

because R is a domain.

$a(x) \cdot b(x) = \overbrace{(a_p b_q)}^{\neq 0} x^{p+q} + \text{strictly higher deg.}$ $\square$

Let $\mathbb{F}$ be a field. Then there are fields

$\mathbb{F}(x) :=$ field of fractions for $\mathbb{F}[x]$.

"R"

$\mathbb{F}((x)) :=$ field of fractions for $\mathbb{F}[[x]]$.

A typical element in a field of fractions is "$\dfrac{a(x)}{b(x)}$" $\in R$

if denominator was invertible in R, then $\in R$.

So how fields of fractions behave is very dependent on which elts of R are invertible.

In $\mathbb{F}[x]$, the <u>only</u> invertible elements are constants.

$\deg(a(x)) \in \mathbb{N} \cup \{-\infty\}$.

<u>Pf:</u> Define the <u>degree</u> of $\sum a_i x^i$ to be the largest d s.t. $a_d \neq 0$.

($\deg(0) := -\infty$). Then $\deg(a(x) \cdot b(x)) = \deg(a(x)) + \deg(b(x))$.

If $a(x) \cdot b(x) = 1$ then $\deg(a) + \deg(b) = 0$.

$\Rightarrow \deg(a) = \deg(b) = 0$. $\square$

$\mathbb{F}(x)$ field of rational functions.

$\Downarrow$

$\dfrac{a(x)}{b(x)}$ for $b(x) \overset{\neq}{0}$ any interesting poly.

In $\mathbb{F}[[x]]$, $(1+x)$ was invertible.

$x + x^2 = x\underbrace{(1+x)}_{\text{inv. in } \mathbb{F}[[x]]}$

In fact, $\sum_{n=0}^{\infty} a_n x^n$ is inv. if $a_0 \neq 0$.

Only "interesting" non-invertible elt is x.

$\Longleftrightarrow$ every nonzero elt is $x^n \cdot$ invertible.

$\mathbb{F}((x)) = \underbrace{\mathbb{F}[[x]][x^{-1}]}_{\text{Laurent series}} \supseteq \underbrace{\mathbb{F}[x][x^{-1}]}_{\text{Laurent polynomials.}}$