MATH 3032: Abstract Algebra

Assignment 1

due 20 January 2023, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with.

1. Let R be a ring, and $n \in \mathbb{Z}$ an integer. Recall that, for every $x \in R$, there is a well-defined element "$n \cdot x$," defined in terms of the additive group structure on R.

   For a positive integer n, we say that R has *characteristic* n if $n \cdot x = 0$ for all $x \in R$ and n is the smallest positive integer with this property. If there does not exist a positive integer n such that $n \cdot x = 0$ for all $x \in R$, then we say that R has *characteristic* 0.

   (a) Show that $\mathbb{Z}_n$ has characteristic n.

   (b) Show that the zero ring is the unique unital ring of characteristic 1.

   (c) Give an example of a nonunital ring of characteristic 1 other than the zero ring.

   (d) Suppose that R is unital, with unit 1_R. Show that R has characteristic n if and only $n \cdot 1_R = 0$.

2. Recall that a ring R is called *Boolean* if for every $x \in R$, $x^2 = x$.

   (a) Show that every Boolean ring is commutative.

   (b) Show that every Boolean ring has characteristic (1 or) 2.

3. The following notion is not normally covered in undergraduate textbooks, but is quite important to some research applications. (For example, it came up in my current research.)

   Let R be a ring. Then R is called *von Neumann regular* (vN regular) if for every $x \in R$, there exists a $y \in R$ such that $xyx = x$.

   (a) Show that every division ring is vN regular.

   (b) Show that every Boolean ring is vN regular.

   (c) Is the zero ring vN regular?

   (d) Is $\mathbb{Z}$ vN regular?

   (e) Is $\mathbb{Z}_{10}$ vN regular?

   (f) Is $\mathbb{Z}_8$ vN regular?

   (g) [Bonus problem — hard!] Is the ring $C(\mathbb{R})$ of continuous functions $\mathbb{R} \to \mathbb{R}$ vN regular?

4. Recall that an *idempotent* in a ring R is an element $p \in R$ such that $p^2 = p$. For example, 0 is an idempotent, and if R is unital, then 1 is also an idempotent. An idempotent other than 0 or 1 is called a *nontrivial idempotent*.

   (a) Show that, if R is a division ring, then all idempotents are trivial.

   (b) Show that, in $\mathbb{Z}$, all idempotents are trivial.

   (c) Find a nontrivial idempotent in $\mathbb{Z}_{15}$. (There are two of them.)

   (d) Suppose that R is commutative and that $p \in R$ is an idempotent. Define subsets $\ker(p) \subset R$ and $\mathrm{im}(p) \subset R$ as follows:

   $$\ker(p) := \{x \in R \text{ s.t. } xp = 0\}, \qquad \mathrm{im}(p) := \{x \in R \text{ s.t. } xp = x\}.$$

   Show that every element $z \in R$ is uniquely expressible as $z = x + y$ with $x \in \ker(p)$ and $y \in \mathrm{im}(p)$.

   (e) Show that $\ker(p)$ and $\mathrm{im}(p)$ are subrings of R.

   (f) Show that $\mathrm{im}(p)$ is unital as a ring. Show that, if R is unital, then $\ker(p)$ is unital as a ring. But show that if p is nontrivial, then neither $\mathrm{im}(p)$ nor $\ker(p)$ is a unital subring of R.

   (g) Show that the function $R \to \mathrm{im}(p)$ sending $x \mapsto xp$ is a ring homomorphism, and that it is a unital ring homomorphism if R is unital.