MATH 3032: Abstract Algebra

Assignment 3

due 28 February 2023, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please be sure to include your surname in the file name.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with.

Remember that all answers should be justified unless otherwise stated.

1. Let R be a *finite* unital ring. Show that every element of R is either a unit or a *left zero-divisor*, an element $a \in R$ such that there exists $b \neq 0$ such that $ab = 0$ [if R is noncommutative, then this might be different from being a right zero-divisor]. Explain why an element cannot be both a unit and a left zero-divisor except for one possible ring R [which one?]. Explain why the main statement implies that in a *finite* unital ring, the set of left zero-divisors is equal to the set of right zero-divisors.

   Hint: Explain that if $r \in R$ is *not* a left zero-divisor if and only if left-multiplication by r is injective. Now use finiteness of R.

2. (a) Does $\mathbb{Z}_4[x]$ contain a non-constant polynomial which is a unit? Either give an example of one or prove that none exists.
   (b) Does $\mathbb{Z}_6[x]$ contain a non-constant polynomial which is a unit? Either give an example of one or prove that none exists.

3. Define the *formal derivative* $\partial_x : R[x] \to R[x]$ to be the operation $\sum_n a_n x^n \mapsto \sum_n n a_n x^{n-1} = \sum_n (n+1) a_{n+1} x^n$.
   (a) Is ∂_x a homomorphism of additive groups? Is ∂ a homomorphism of rings?
   (b) What is the kernel of $\partial_x : \mathbb{Z}[x] \to \mathbb{Z}[x]$?
   (c) What is the kernel of $\partial_x : \mathbb{Z}_p[x] \to \mathbb{Z}_p[x]$?
   (d) What is its image of $\partial_x : \mathbb{Z}_p[x] \to \mathbb{Z}_p[x]$?

4. For each of the following pairs $f, g \in R[x]$, use long division to write $f = qg + r$ with $\deg r < \deg g$. You should do the work by hand and show your work, but you do not need to write any words of explanation.
   (a) $f(x) = x^6 + 3x^5 + 4x^2 - 3x + 2$ and $g(x) = x^2 + 2x - 3$ in $\mathbb{Z}[x]$
   (b) $f(x) = x^6 + 3x^5 + 4x^2 - 3x + 2$ and $g(x) = 3x^2 + 2x - 3$ in $\mathbb{Z}_7[x]$.