Math 3032: Abstract Algebra

Assignment 5

Solutions

1. **Find a Gröbner basis for the ideal $\langle x^2y - x - 2, xy + 2y - 9\rangle \subset \mathbb{R}[x, y]$ with respect to the ordering $x \gg y$. Describe, in as much detail as you can, the corresponding algebraic variety, i.e. the set of solutions to the system of equations $x^2y - x - 2 = xy + 2y - 9 = 0$.**

   We start with the basis listed on the top row. Each cell starts with our current basis element and then describes a replacement operation on it. So at any given time, the basis consists of those elements at the top of the cells for that row.

$$
\begin{array}{c|c}
x^2y - x - 2 & xy + 2y - 9 \\
-x(xy + 2y - 9) & \\
\cline{1-1}
-2xy + 8x - 2 & \\
\times(-\frac{1}{2}) & \\
\cline{1-1}
xy - 4x + 1 & \\
-(xy + 2y - 9) & \\
\cline{1-1}
-4x - 2y + 10 & \\
\times(-\frac{1}{2}) & \\
\cline{1-1}
2x + y - 5 & \\
 & \times 2 \\
\cline{2-2}
 & 2xy + 4y - 18 \\
 & -y(2x + y - 5) \\
\cline{2-2}
 & -y^2 + 9y - 18 \\
 & \times(-1) \\
\cline{2-2}
 & y^2 - 9y + 18
\end{array}
$$

   At this point our basis consists is $\langle 2x + y - 5, y^2 - 9y + 18\rangle$. This is a Gröbner basis because we showed in class that any basis of the form $\langle x + p(y), q(y)\rangle$ is Gröbner.

   To solve the corresponding system of equations, we first note that $y^2 - 9y + 18 = (y - 3)(y - 6)$, and so if $y^2 - 9y + 18 = 0$, then $y = 3$ or $y = 6$. In the first case, if $2x + y - 5 = 0$, then $x = 1$, and in the second case $x = -\frac{1}{2}$. So the solutions are $\{(1, 3), (-\frac{1}{2}, 6)\}$.

2. **Find a Gröbner basis for the ideal $\langle x^2y + x + 1, xy^2 + y - 1\rangle \subset \mathbb{R}[x, y]$ with respect to the ordering $x \gg y$. Describe, in as much detail as you can, the corresponding algebraic variety, i.e. the set of solutions to the system of equations $x^2y + x + 1 = xy^2 + y - 1 = 0$.**

   There is no valid division available: $x^2y \gg xy^2$, but x^2y is not divisible by xy^2. There is a chance that the basis is already Gröbner, and to find out, or to proceed with the algorithm, we must consider the polynomial

$$s(x, y) = y(x^2y + x + 1) - x(xy^2 + y - 1) = xy + y - xy + x = x + y.$$

Since this is smaller than any of our original basis vectors, the original basis was not Gröbner. Instead, we extend our basis, and work with $\langle x^2y + x + 1, xy^2 + y - 1, x + y\rangle$. With notation as above, we have:

$$
\begin{array}{c|c|c}
x^2y + x + 1 & xy^2 + y - 1 & x + y \\
-xy(x + y) & & \\ \cline{1-1}
-xy^2 + x + 1 & & \\
+(xy^2 + y - 1) & & \\ \cline{1-1}
x + y & & \\
-(x + y) & & \\ \cline{1-1}
0 & & \\
 & -y^2(x + y) & \\ \cline{2-2}
 & -y^3 + y - 1 & \\
 & \times(-1) & \\ \cline{2-2}
 & y^3 - y + 1 &
\end{array}
$$

We remove 0's from the basis. Our basis is $\langle x + y, y^3 - y + 1\rangle$. This basis is Gröbner for the reason mentioned above.

The equation $y^3 - y + 1 = 0$ has one real solution (Proof: The derivative $3y^2 - 1$ vanishes at $y = \pm\frac{1}{\sqrt{3}}$, but since $\frac{1}{\sqrt{3}} < 1$, the local minimum at $y = +\frac{1}{\sqrt{3}}$ gives a positive value to $y^3 - y + 1$. So the graph of the function $y \mapsto y^3 - y + 1$ starts very negative when $y \to -\infty$, then as y increases the graph goes up, crosses through 0, turns around, turns around again *before* crossing zero, and then continues on to $+\infty$), and two imaginary solutions. All three solutions are irrational (Proof: Any rational solution is integral, since the equation is monic, but any integral solution would divide 1, and neither ± 1 is a solution). For each solution, the equation $x + y = 0$ has a corresponding solution. In other words, the solutions to the original system are the pairs $(x, y) = (-\alpha, \alpha)$ where $y = \alpha$ is a solution to $y^3 - y + 1 = 0$.