Math 3032: Abstract Algebra

Assignment 7

due April 11, 2023

Recall that the power set $\mathcal{P}(\mathbb{N}) = \{\text{subsets of } \mathbb{N}\}$ is a Boolean ring, with addition and multiplication defined by

$$A + B := (A \cup B) \smallsetminus (A \cap B), \qquad A \cdot B = A \cap B.$$

Pick once and for all a non-principal maximal ideal $\mathfrak{m} \subset \mathcal{P}(\mathbb{N})$. Let's refer to the elements of $\mathfrak{m}$ as *minorities* and the elements of $\mathcal{P}(\mathbb{N}) \smallsetminus \mathfrak{m}$ as *majorities*. Here are a few facts that you should think through, but you don't need to write any justification if you don't want to:

- The multiplicative unit "1" in $\mathcal{P}(\mathbb{N})$ is $\mathbb{N}$ itself. For any element $A \in \mathcal{P}(\mathbb{N})$, its complement is $\neg A := \mathbb{N} \smallsetminus A = 1 + A$.
- The requirement that $\mathfrak{m}$ is an ideal is equivalent to the requirements that: (i) if you expand a majority, then it remains a majority; (ii) the intersection of two majorities is again a majority.
- The requirement that $\mathfrak{m}$ is maximal is equivalent to the requirement that, for every element $A \in \mathcal{P}(\mathbb{N})$, either A or $\neg A$ is in $\mathfrak{m}$, but not both.
- The requirement that $\mathfrak{m}$ is non-principal is equivalent to the requirement that if $A \subset \mathbb{N}$ is *finite*, then A is a minority.

Now consider the ring $\mathbb{R}^{\mathbb{N}}$ of all functions $\mathbb{N} \to \mathbb{R}$. We'll equivalently think of a function $a : \mathbb{N} \to \mathbb{R}$ as the sequence $(a(0), a(1), a(2), \dots)$. Addition and multiplication are defined pointwise (using the addition and multiplication in $\mathbb{R}$):

$$(a + b)(n) := a(n) + b(n), \quad (a \cdot b)(n) := a(n) \cdot b(n).$$

Let $J_{\mathfrak{m}} \subset \mathbb{R}^{\mathbb{N}}$ denote the set of sequences for which a majority of its entries are 0. (Note that this depends on the choice of $\mathfrak{m}$.)

1. Show that $J_{\mathfrak{m}}$ is an ideal. Define ${}^*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/J_{\mathfrak{m}}$ to be the quotient ring. Thus, an element of ${}^*\mathbb{R}$ is represented by a sequence $f : \mathbb{N} \to \mathbb{R}$. Explain that two sequences represent the same element iff they agree on a majority of entries. Explain that if you want to describe an element of ${}^*\mathbb{R}$, you don't have to describe all values of a representing sequence: just describing a majority of its entries suffices. The ring ${}^*\mathbb{R}$ is called the ring of *hyperreal numbers*. (Warning: different choices of $\mathfrak{m}$ can give non-isomorphic rings, so it really should be called the "$\mathfrak{m}$-hyperreal numbers.")
2. Show that ${}^*\mathbb{R}$ is a field. Hint: For any sequence, either a majority of its entries are 0, or a majority of its entries are invertible.

3. Show that, if $x \in {}^*\mathbb{R}$, then exactly one of the following is true: x is 0, or x is a nonzero square, or $-x$ is a nonzero square. Let's call x *positive* if x is a nonzero square, and *negative* if $-x$ is a nonzero square. Hint: for any sequence, either a majority of its entries are zero, or a majority of its entries are positive, or a majority of its entries are negative.

4. Show that the relation "$x < y$ if $y - x$ is positive" is a total ordering on ${}^*\mathbb{R}$. Hint: the thing you need to prove is that the sum of two positive elements is positive (why?). Hint: use the fact that the intersection in $\mathbb{N}$ of majorities is again a majority.

5. The map $\mathbb{R} \to \mathbb{R}^{\mathbb{N}} \to {}^*\mathbb{R}$ that takes each real number r to the constant sequence $(r, r, r, r, \dots)$ is a homomorphism (why?). Hence it is an injection (why?). Let's call this homomorphism $r \mapsto {}^*r$. Let's call those hyperreals in the image of this map *standard*.

   Let's say that a hyperreal $x \in {}^*\mathbb{R}$ is *positive infinite* if it is greater than every standard real number. Show that positive infinite hyperreals exist. Hint: consider an unbounded increasing sequence, for example $1, 2, 3, 4, \dots$.

   Let's say that a hyperreal x is *infinite* if x or $-x$ is positive infinite. Let's say that a hyperreal is *finite* if it is not infinite.

6. Show that the set ${}^*\mathbb{R}_{<\infty}$ of finite hyperreals is a ring.

7. Let's say that a hyperreal x is *infinitesimal* if for every positive real number $r \in \mathbb{R}_{>0}$, $x < r$ and $-x < r$. For example, 0 is infinitesimal. Show that nonzero infinitesimals exist. Hint: consider a sequence that converges to 0, but never gets there.

8. Show that the set ${}^*\mathbb{R}_{\approx 0}$ of infinitesimal hyperreals is an ideal in the ring of finite hyperreals. Let's say that two finite hyperreals are *approximately equal* if their difference is infinitesimal. Since ${}^*\mathbb{R}_{\approx 0}$ is an ideal, approximate equality is an equivalence relation, and two finite hyperreals are approximately equal iff they represent the same class in the quotient ring ${}^*\mathbb{R}_{<\infty}/{}^*\mathbb{R}_{\approx 0}$. For a finite hyperreal x, the set of all hyperreals approximately equal to x is called its *halo*.

9. Show that if r, s are standard, then ${}^*r \approx {}^*s$ if and only if $r = s$. Hard (requires real analysis): Show that every finite hyperreal x is approximately equal to some standard real ${}^*r$. This standard number r is called the *body* of x, and $x - \operatorname{body}(x)$ is called the *soul* of x.

   Conclude that the quotient ring ${}^*\mathbb{R}_{<\infty}/{}^*\mathbb{R}_{\approx 0}$ is isomorphic to $\mathbb{R}$.

Hyperreals were invented in the 1960s by Abraham Robinson, and described in detail in his book *Nonstandard analysis* (1966). His idea was that a lot of real analysis can be done without ever using limits and ϵ-δ proofs and keeping track of bounds and errors. For example, if you have some function $f : \mathbb{R} \to \mathbb{R}$, then you can extend it to $\mathbb{R}^{\mathbb{N}}$ pointwise (in other words, $f(a_0, a_1, a_2, \dots) = (f(a_0), f(a_1), f(a_2), \dots))$. Now, if sequences a and b agree at a majority of entries, then certainly $f(a)$ and $f(b)$ also agree at a majority of entries. So f defines a function ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$. Now, suppose that f has some analytic behaviour you care about. For example, maybe f is bounded, i.e. there are some real numbers $R_-, R_+ \in \mathbb{R}$ such that $R_- < f(r) < R_+$ for all $r \in \mathbb{R}$. Then these also bound ${}^*f(x)$ for every hyperreal x. But then you can pick some infinite hyperreal ω, and look at $\operatorname{body}(f(\omega))$, and this is a sort of "$\lim_{x\to\infty} f(x)$". If you wanted a limit to 0 instead, you could pick some nonzero infinitesimal hyperreal ϵ and look at $\operatorname{body}(f(\epsilon))$. Maybe you want to know which of two functions f, g "grows faster": just ask whether $f(\omega)$ or $g(\omega)$ is bigger. Etc.