Math 3032:

Announcements/suggestions:

- switch to T Th 11:30 – 1?
- I posted solns to HW 1.

R a comm unital ring. An ideal $J \subseteq R$ is like a $\underbrace{\text{number}}_{\in R}$ up to units. Numbers for purpose of div.

$$\{\text{ideals}\} \supseteq \{\text{principal ideals}\} = R/R^\times$$

- For $\mathbb{Z}$: this $\supseteq$ is $=$
- For $\mathbb{Z}[\sqrt{-5}]$, this $\supseteq$ is $\supsetneq$.

Driving question: which ideal are prime?

We also said: Ideals are the things you can quotient by to get another ring:

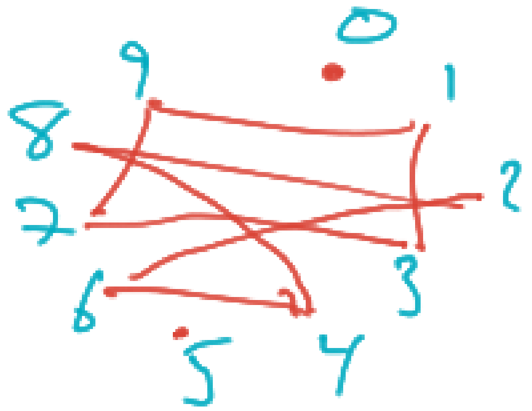
- if R is a ring, $J \subseteq R$ an ideal, then R/J is a ring.

Who are the ideals in R/J?

E.g.: $R = \mathbb{Z}$, $J = (10)$. $R/J = \mathbb{Z}_{10}$.

Since every subgp of a cyclic gp is cyclic, every ideal in $\mathbb{Z}_{10}$ is principal.

$(\mathbb{Z}_{10})^\times = \underbrace{1, 3, 7, 9}$ mod 10.



Ideals in $\mathbb{Z}_{10}$: (10), (1), (2), (5).

In the example, ideals in $\mathbb{Z}/(10)$ $\equiv$ divisors of 10.

In ideals, "n divides 10" $\equiv$ $(n) \supseteq (10)$.

__Claim:__ For any ring R and ideal J, the ideals in R/J are indexed by the ideals $I \supseteq J$.

__Pf:__ Suppose $K \subseteq R/J$ is an ideal.

$$\begin{tikzcd} K \arrow[r, phantom, "\subseteq"] & R/J \\ I \arrow[u, dashed] \arrow[r, phantom, "\subseteq"] & R \arrow[u, "{\text{canonical.} \; r \mapsto r+J=[r]}"'] \end{tikzcd}$$

Define $I \subseteq R$ by $I = \{r \in R \text{ s.t. } [r] \in K\}$

<u>subclaim:</u>

- $I \subseteq R$ is an ideal
- $I \supseteq J$. $\Leftarrow$ completely obvious.

suppose $x \in I$, and $y \in R$.

WTS: $xy \in I$.

<u>Equiv:</u> $[xy] \in K$.

<u>But:</u> $[xy] = \underset{K}{[x]} \cdot \underset{R/J}{[y]}$

Then use that $K \subseteq R/J$ is an ideal.

$\{\text{ideals in } R/J\} \rightsquigarrow \{\text{ideals between } J \text{ and } R\}$.

fixed Input: $J \subseteq R$

"between": $J \subseteq I \subseteq R$

Suppose I have any additive subgp $R \supseteq I \supseteq J$

Then I can look at $I/J \subseteq R/J$ subgp.

Second half of claim:

if $I \subseteq R$ is an ideal containing J,

then $I/J \subseteq R/J$ is an ideal. Pretty obv.

i.e. WTS: $[y] \in I/J$ and $[x] \in R/J$ then $[x]\cdot[y] \in I/J$ ✓

i.e. $y \in I$ and $x \in R$ $\quad [xy]$ WTS: $xy \in I$.

$$\{\text{ideals } I \text{ w/ } J \subseteq I \subseteq R\} \rightleftarrows \{\text{ideals in } R/J\}$$

$$I \longmapsto I/J.$$

$$\{r \in R \text{ s.t. } [r] \in K\} \longmapsfrom K$$

Last thing to check: $\circlearrowleft$ and $\circlearrowright$ are ids.

Follows from the corresponding statement for gps.

One of the reasons for ideals was to form quotient rings.

One of the "isomorphism theorems" for rings:

If R is a ring, $J \subseteq R$ an ideal, and $I \subseteq R$ an ideal w/ $I \supseteq J$ then

$$\frac{R/J}{I/J} \cong R/I$$

One of our intuitions for "prime":

p is prime if p is not trivial (not unit) but all divisors of p are.

In terms of ideals:

J s.t. if $I \supseteq J$ then $I = J$ or $I = R$.

These are the <u>maximal</u> ideal's.

<u>Equiv to</u>: the quotient ring $R/J = F$ has no nontrivial ideals, i.e. its only ideals in F are $\{0\}$ and F.

Which unital comm. rings have no nontriv ideals?

Suppose F is a ring, pick $x \in F$, $x \neq 0$.

Then (x) is an ideal, not $\{0\}$.

If F has no nontriv. ideals, then $(x) = F$.

But $1 \in F$. So $1 \in (x)$. So $\exists y$ s.t. $xy = 1$.

So x is invertible.

So F is a field.

Conv: If F is a field and $I \subseteq F$ an ideal.

Either $I = \{0\}$, or $I \ni x \neq 0$. So $I \ni x x^{-1}$.

So $I = F$.

Fields $\equiv$ unital com rings (not the zero ring) w/ no nontrivial ideals.

$\equiv$ unital com rings w/ exactly two ideals.

<u>Prop:</u> For any unital com ring R, an ideal $J \subseteq R$ is maximal iff R/J is a field.

Another notion of prime:

p is <u>prime</u> if:

whenever p divides ab,

p necessarily divides at least one of a, b.

<u>Defn:</u> An ideal $J \subseteq R$ is <u>prime</u> if:

whenever $\underbrace{ab \in J}$, at least one of a or $b \in J$.

$\Updownarrow$

$[a]\cdot[b] = [ab] = 0$ in R/J

Prop:

$J \subseteq R$ is prime iff

$\forall [a], [b] \in R/J$, if $[a]\cdot[b] = 0$, then at least one of $[a], [b] = 0$.

i.e. R/J is an integral domain.

E.g.: (0) is prime iff R is an integral domain.