Math 3032: Abstract Algebra

Midterm exam - Solutions

7 March 2023

Part A.

1. **State the distributive law in ring theory.**

   In a ring R, every $x, y, z \in R$,

   $$x(y+z) = xy + xz \quad \text{and} \quad (x+y)z = xz + yz.$$

   Note: In a commutative ring, one of these implies the other, but not in a noncommutative ring.

2. **Give an example of a unique factorization domain which is not a principal ideal domain.**

   $\mathbb{C}[x, y]$ and $\mathbb{Z}[x]$ are both examples. These are UFDs because $\mathbb{C}[x]$ and $\mathbb{Z}$ are EDs, and hence UFDs, and in general if R is a UFD, then so is $R[x]$. They are not PIDs because in general, if R is an integral domain and $r \in R$ is not zero or invertible, then the ideal $(r, x) \in R[x]$ is not principal (since otherwise its generator would have to be a constant, so that it could divide r but then it would have to be invertible, so that it could divide x, but the only constants in (r, x) are the multiples of r).

3. **Consider the following statement:**

   > **If R is a commutative ring and $J \subset R$ is an ideal, then there exists a commutative ring homomorphism $\varphi : R \to S$ with $\ker(\varphi) = J$.**

   Either show that this statement is true by giving an example of such a homomorphism φ, or show that this statement is false by giving an example of a ring R with an ideal J for which no such homomorphism exists.

   Take $S = R/J$ the quotient ring, and $\varphi : R \to S$ the canonical homomorphism sending $r \mapsto r + J$. The Isomorphism Theorems state that this map is a homomorphism with kernel J.

4. **Is there a field F and an injective ring homomorphism $\mathbb{Z}_6 \hookrightarrow F$? If so, describe such an F, and if not, explain why not.**

   No, since any subring of a field is necessarily an integral domain, but $\mathbb{Z}_6$ is not (since $2 \cdot 3 = 0$ in $\mathbb{Z}_6$).

Part B.

Prove that $x^3 + 2x^2 + 3 \in \mathbb{Q}[x]$ is irreducible.

A monic (more generally, a primitive) integral polynomial is irreducible over $\mathbb{Q}$ if and only if it is irreducible over $\mathbb{Z}$. For cubics, any factorization must contain a linear factor; for monics, any factorization must be into monics. Thus $x^3 + 2x^2 + 3 \in \mathbb{Q}[x]$ is irreducible if and only if it has no integral roots.

If $x \geq 0$, then $x^3 + 2x^2 + 3 \geq 3$, so no nonnegative integers are roots.

Suppose that $x = -y$ with $y > 3$. Then $-(x^3 + 2x^2 + 3) = y^3 - 2y^2 - 3$. Now, since $y > 3$, $y^3 - 2y^2 = (y - 2)y^2 > (3 - 2)y^2 = y^2$. But since $y > 3$, $y^2 - 3 > 0$. So there are no roots with $x < -3$.

So we need only to check the values $x = -3, -2, -1$.

$(-3)^3 + 2(-3)^2 + 3 = -27 + 18 + 3 = -6.$

$(-2)^3 + 2(-2)^2 + 3 = -8 + 8 + 3 = 3.$

$(-1)^3 + 2(-1)^2 + 3 = -1 + 2 + 3 = 4.$

So there are no integral roots.