Math 3032: Abstract Algebra

Practice final

24 April 2023

Your name:

University academic honour statement:

Dalhousie University has adopted the following statement, based on "The Fundamental Values of Academic Integrity" developed by the International Center for Academic Integrity (ICAI):

> Academic integrity is a commitment to the values of learning in an academic environment. These values include honesty, trust, fairness, responsibility, and respect. All members of the Dalhousie community must acknowledge that academic integrity is fundamental to the value and credibility of academic work and inquiry. We must seek to uphold academic integrity through our actions and behaviours in all our learning environments, our research, and our service.

Please **sign here** to confirm that you will uphold these values, and that the work you submit on this exam will be your own.

Exam structure

There are six questions, each worth ten points.

Question 1.

Suppose that R is a unital commutative ring. State the definition of *zero-divisor* in R. Prove that $r \in R$ is *not* a zero-divisor if and only if the function $r \times (-) : R \to R$ is injective.

Question 2.

Suppose that $f : R \to S$ is a surjective ring homomorphism, and that R is unital. Show that S is also unital.

Question 3.

When is an element of a ring called *irreducible*? When is it called *prime*? Give an example of an element of a ring which is irreducible but not prime. Give an example of an element of a ring which is prime but not irreducible.

Question 4.

Draw a picture of the ideal $\langle\sqrt{-3}\rangle \subset \mathbb{Z}[\frac{1-\sqrt{-3}}{2}]$.

Question 5.

Let R be a commutative ring. Recall that an element $r \in R$ is *nilpotent* ("zero-powered") if $r^n = 0$ for some $n \in \mathbb{N}$. Prove that the set of nilpotent elements is an ideal in R.

Question 6.

Run Buchberger's algorithm to find a Gröbner basis for the ideal $\langle x^2y + x, xy^2 - y \rangle \subset \mathbb{R}[x, y]$ with respect to the ordering $x \gg y$.