Math 3032: Abstract Algebra

Practice Midterm Solutions

7 March 2023

Part A.

1. **Suppose that $f(x), g(x) \in \mathbb{Z}_3[x]$, and that, by performing long division, you write $f(x) = q(x)g(x) + r(x)$. The ideal $(f(x), g(x))$ is necessarily equal to one of the following:**

$$(f(x), r(x)), \qquad (g(x), r(x)), \qquad (f(x), q(x)), \qquad (g(x), q(x)), \qquad (q(x), r(x))$$

Which one?

The correct answer is $\boxed{(g(x), r(x))}$. In general, the ideal (f, g) will be equal to (a, b) iff $a, b \in (f, g)$ and $f, g \in (a, b)$. The ideal (f, g) does not necessarily contain q, but does necessarily contain r, so we can rule out the last three answers. The ideal (f, r) does not necessarily contain g (it merely contains a multiple of g), but the ideal (g, r) does necessarily contain f. So the last answer is correct.

2. **In the following sentence, should the blank be filled in with the word "all," the words "some but not all," or the words "none of the"?**

In the ring $\mathbb{C}[x, y]$, ____________ maximal ideals are principal.

You do not need to justify your answer.

The correct answer is $\boxed{\text{none of the}}$. Suppose that $\mathfrak{m} \subset \mathbb{C}[x, y]$ is maximal. Since $\mathbb{C}$ is algebraically closed, $\mathbb{C}[x, y]/\mathfrak{m}$ is isomorphic to $\mathbb{C}$ (via a unique isomorphism which is the identity on $\mathbb{C} \subset \mathbb{C}[x, y]$), and so there exist constants $a, b \in \mathbb{C}$ such that the homomorphism $\mathbb{C}[x, y] \to \mathbb{C}[x, y]/\mathfrak{m} \cong \mathbb{C}$ sends $x \mapsto a$ and $y \mapsto b$. So $x - a$ and $y - b$ are both in $\mathfrak{m}$, and so $\mathfrak{m} \neq (0)$. Moreover, if $\mathfrak{m}$ were principal, then its generator would have to have degree at most 0 in x (since it divides $y - b$) and also at most 0 in y (since it divides $x - a$), and so must be a (nonzero!) constant; but nonzero constants are invertible, and so $\mathfrak{m}$ would be the whole ring, which is forbidden by the word "maximal." Thus $\mathfrak{m}$ is not principal.

3. **If $\varphi : R \to S$ is a ring homomorphism, is it necessarily true that $S \cong R/\ker(\varphi)$? If no, what extra condition on φ is needed to make it true?**

   No. The Isomorphism Theorems only supply such an isomorphism when φ is $\boxed{\text{surjective}}$.

4. **Suppose that F is a field and $\varphi : R \to F$ is a ring homomorphism. Which of the following statements are necessarily true? You do not need to justify your answers.**

   - **If φ is injective, then R is commutative.** $\boxed{\text{TRUE}}$
   - **If φ is injective, then R is an integral domain.** $\boxed{\text{TRUE}}$
   - **If φ is injective, then the ideal $\ker(\varphi)$ is maximal.** $\boxed{\text{FALSE}}$
   - **If φ is surjective, then the ideal $\ker(\varphi)$ is maximal.** $\boxed{\text{TRUE}}$
   - **If φ is surjective, then F is isomorphic to the field of fractions of R.** $\boxed{\text{FALSE}}$

Part B.

Prove that $x^4 - 4x^3 + 6x^2 + 11x + 6 \in \mathbb{Q}[x]$ **is irreducible. Hint: substitute** $x \mapsto x+1$**.**

Note that $f(x) \in \mathbb{Q}[x]$ is irreducible iff $f(x+1)$ is. Following the hint (or noting that $x^4 - 4x^3 + 6x^2 = (x-1)^4 +$ small terms), we substitute:

$$
\begin{aligned}
&(x+1)^4 - 4(x+1)^3 + 6(x+1)^2 + 11(x+1) + 6 \\
&\quad = (x^4 + 4x^3 + 6x^2 + 4x + 1) - 4(x^3 + 3x^2 + 3x + 1) + 6(x^2 + 2x + 1) + 11(x+1) + 6 \\
&\quad = x^4 + 4x^3 - 4x^2 + 6x^2 - 12x^2 + 6x^2 + 4x - 12x + 12x + 11x + 1 - 4 + 6 + 11 + 6 \\
&\quad = x^4 + 0x^3 + 0x^2 + 15x + 20.
\end{aligned}
$$

Eistenstein's criterion with $p = 5$ completes the proof.