MATH 4180/5180: Algebraic Topology

Assignment 1

due January 23, 2023

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. Show that a retract of a contractible space is contractible.

2. Let $X \subset \mathbb{R}^3$ denote the image of the standard immersion of a Klein bottle. Show that X is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

3. Show that $\vee$ is the coproduct of pointed spaces (i.e. in the category of pointed spaces and pointed continuous maps).

   Show that $\wedge$ is the *tensor product* of pointed spaces in sense of hom-tensor adjunctions: given pointed spaces Y, Z, the set $\hom_*(Y, Z)$ of pointed maps from Y to Z is naturally a space; the statement to be proven is that $\hom_*(X \wedge Y, Z) = \hom_*(X, \hom_*(Y, Z))$.

   Remark: Since we haven't talked at all about point-set issues, you should use without proof that $\times$ is the tensor product, in the same sense, in the category of unpointed spaces.

4. Show that S^∞ is contractible.

5. Given positive integers v, e, f with $v - e + f = 2$, find a cell decomposition of S^2 with v 0-cells, e 1-cells, and f 2-cells.

6. A map $X \to Y$ is *constant* if it factors as $X \to \{\mathrm{pt}\} \to Y$. A map is *homotically constant* if it is homotopic to a constant map. Show that the following are equivalent for a space X:

   (a) X is contractible.

   (b) For every space Y, every map $X \to Y$ is homotopically constant.

   (c) For every space Y, every map $Y \to X$ is homotopically constant.

7. Suppose that $f : X \to Y$ and there exists $g, h : Y \to X$, possibly different, such that $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$. Show that f is a homotopy equivalence.

8. The *join* $X * Y$ of spaces X and Y is the quotient of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ for any fixed $x \in X$ and any two $y_1, y_2 \in Y$, and $(x_1, y, 1) \sim (x_2, y, 1)$ for any fixed $y \in Y$ and any two $x_1, x_2 \in X$.

   (a) Show that the cone CX is $X * \{\mathrm{pt}\}$.

   (b) Show that the suspension SX is $X * S^0$.

   (c) Show that $S^m * S^n = S^{m+n+1}$.