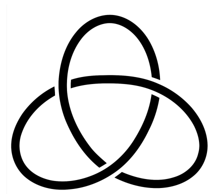
MATH 4180/5180: Algebraic Topology

Assignment 2: π_1

due February 28, 2023

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$. Hint: Use π_1 to show that they are non-homotopic.
2. Let $X \subset \mathbb{R}^m$ be a union of convex open subsets X_i, such that for every triple i, j, k, the triple intersection $X_i \cap X_j \cap X_k$ is nonempty. Show that X is simply connected.
3. Let M be a manifold of dimension $n \geq 3$, and suppose that M is simply connected. Show that M remains simply connected if you remove finitely many points.
4. Let X denote the complement of a trefoil knot in $\mathbb{R}^3$. Pick a basepoint $x_0 \in X$ "above" the knot, with respect to the standard drawing of the trefoil:

   Note that, in this standard drawing, there are three "arcs" that go over but not under other strands. For each of these arcs, consider the loop in X that leaves from x_0, loops just around that arc, and then heads back to the basepoint, without interacting with the other strands. This selects three elements $a, b, c \in \pi_1(X, x_0)$.

   (a) Show that a, b, c together generate $\pi_1(X, x_0)$.
   (b) Show that a, b, c satisfy the relations $aba^{-1} = c$, $bcb^{-1} = a$, $cac^{-1} = b$. Show that these relations generate all relations.
   (c) Construct a nontrivial surjection $\pi_1(X, x_0) \to S_3$, the symmetric group on three elements. Conclude that $\pi_1(X, x_0)$ is nonabelian. Conclude that the trefoil is nontrivially knotted.

   Remark: You may use van Kampen's theorem, but also feel free to argue slightly informally about how loops can interact.
5. Recall that the *reduced suspension* of a pointed space (X, x_0) is $\Sigma(X, x_0) = SX/S\{x_0\}$; it is pointed by the image of x_0. Explain that there is a *loops-suspension adjunction*: for any pointed spaces (X, x_0) and (Y, y_0), show that the sets of pointed maps

   $$\mathrm{maps}_*((X, x_0), \Omega(Y, y_0)), \qquad \mathrm{maps}_*(\Sigma(X, x_0), (Y, y_0))$$

   are naturally isomorphic.

6. Show that if a connected space X has finite π_1, then every map $X \to S^1$ is nullhomotopic (=homotopic to a constant map). Hint: use the covering space $\mathbb{R}^1 \to S^1$.
7. Find all 2- and 3-sheeted covers of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoint.
8. Find all connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.