MATH 4180/5180: Algebraic Topology

Assignment 3: $\mathrm{H}_\bullet$

due March 17, 2023

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. Determine whether there exists a short exact sequence

$$0 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0.$$

   More generally, determined which abelian groups A fit into a short exact sequence

$$0 \to \mathbb{Z}/p^m \to A \to \mathbb{Z}/p^n \to 0.$$

   What about the case

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n \to 0?$$

2. Show that $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_{n+1}(SX)$ for all n, where SX is the suspension of X. Construct an explicit chain map $s : \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(SX)$ inducing the isomorphism $\widetilde{\mathrm{H}}_n(X) \xrightarrow{\sim} \widetilde{\mathrm{H}}_{n+1}(SX)$.

3. The *degree* of a map $S^n \to S^n$ is by definition the induced map $\mathbb{Z} \cong \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n) \cong \mathbb{Z}$. Suppose that $f(z)$ is a polynomial with complex coefficients, considered as a map $f : \mathbb{C} \to \mathbb{C}$. Recall that $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \cong S^2$, and note that f extends (uniquely) to a map $\mathbb{CP}^1 \to \mathbb{CP}^1$. Show that the degree of f in the sense of topology is equal to the degree of f in the sense of polynomials.

4. Show that the quotient map $S^1 \times S^1 \to S^1 \wedge S^1 = S^2$ is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

5. Let A be a finitely generated abelian group, and let A_{tor} be its subgroup of those elements of finite order. Then $A/A_{\mathrm{tor}} =: A_{\mathrm{free}}$ is a free abelian group, and A factors (noncanonically) as a direct sum $A_{\mathrm{tor}} \oplus A_{\mathrm{free}}$. The *rank* $\mathrm{rk}(A)$ is by definition the rank of A_{free}.

   (a) Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then $\mathrm{rk}(B) = \mathrm{rk}(A) + \mathrm{rk}(C)$.

   (b) Conclude that if $C_\bullet$ is a finitely-generated chain complex (i.e. the total group $\bigoplus_i C_i$ is finitely generated), then

$$\chi(C_\bullet) := \sum (-1)^i \,\mathrm{rk}(C_i) = \sum (-1)^i \,\mathrm{rk}(\mathrm{H}_i(C_\bullet)).$$

   The number $\chi(C_\bullet) = \chi(\mathrm{H}_\bullet(C_\bullet))$ is called the *Euler characteristic* of $C_\bullet$. If X is a space whose total homology is finitely generated, then $\chi(X) := \chi(\mathrm{H}_\bullet(X))$.

(c) What is the Euler characteristic of a genus-g surface? What about $\mathbb{RP}^2$? A Klein bottle? $\mathbb{RP}^n$? $\mathbb{CP}^n$?

(d) Show that if X, Y are finite CW complexes, then $\chi(X)$ and $\chi(Y)$ are defined, and that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(e) Show that if X is a finite CW complex which is a union of subcomplexes A and B, then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$. In particular, χ takes disjoint unions to sums.

(f) Show that, if X is a finite CW complex and $Y \to X$ is an n-sheeted cover, then $\chi(Y) = n\chi(X)$.

(g) Using question 2, how does χ behave under suspension?

6. Using question 5f, show that the only nontrivial finite group that can act freely on an even-dimensional sphere is $\mathbb{Z}/2$.

7. Compute the homology of a Klein bottle.

   Now go to `https://chat.openai.com/` and ask ChatGPT to compute the homology of a Klein bottle. It will probably get it wrong. When it does, have a conversation with it, and see if you can get it to correct its mistakes.

   In the unlikely case that ChatGPT gets the question right, try asking it question 6, or some other questions from the class.

   Note: ChatGPT *does not include a logic engine.* It tries to answer math questions just by pattern-matching the language, and it tends to agree with whatever you tell it.