MATH 4180/5180: Algebraic Topology

Assignment 4: $\mathrm{H}^\bullet$

due March 30, 2023

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. Let $k, \ell > 0$. Show that every map $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial map $\mathrm{H}_{k+\ell}(S^{k+\ell}) \to \mathrm{H}_{k+\ell}(S^k \times S^\ell)$. Hint: dualize to cohomology, and use cup products.

2. Calculate the cohomology ring (with $\mathbb{Z}$ coefficient, say), of a (closed, oriented) surface of genus g. Hint: by attaching $g-1$ 2-cells, you can create a CW complex homotopy-equivalent to a wedge sum of g copies of $T^2 = S^1 \times S^1$.

3. Show that if $\mathrm{H}_n(X)$ is a finitely generated free abelian group for each n, then the rings $\mathrm{H}^\bullet(X;\mathbb{Z}) \otimes \mathbb{Z}/p$ and $\mathrm{H}^\bullet(X;\mathbb{Z}/p)$ are isomorphic. Show that the conclusion holds more generally if $\mathrm{H}_n(X)$ is finitely generated and all elements of finite order have order coprime to p.

4. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

5. Fix a field F. The *Poincaré series* of a space X is the formal power series $p(X)(t) = \sum_{i=0}^\infty \dim_F \mathrm{H}^i(X;F)t^n$. Show that $p(X \times Y) = p(X)p(Y)$ and that $p(X \sqcup Y) = p(X) + p(Y)$. Compute the Poincaré series of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$.