MATH 4180/5180: Algebraic Topology

Optional Assignment 5: $\pi_\bullet$

1. (a) Let (X, x_0) be an H-space, with multiplication $\mu : X \times X \to X$ (and identity element x_0). Consider the operation on $\pi_n(X, x_0)$ defined by

$$[f] \star [g] = [\mu(f, g)], \quad f, g : (D^n, \partial D^n) \to (X, x_0),$$

in other words, $\star$ is the map that μ induces on π_n. Show that $\star$ agrees with the usual group operation on π_n.

   (b) Conclude that $\pi_1(X, x_0)$ is abelian.

   (c) Generalize this argument to show that all Whitehead brackets vanish.

2. Let X be a pointed connected space and $X^{\mathrm{sc}} \to X$ its simply connected cover. Recall that the induced map $\pi_n(X^{\mathrm{sc}}) \to \pi_n(X)$ is an isomorphism for $n \geq 2$, and recall that $\pi_1(X)$ acts on X^{sc}, and hence on $\pi_n(X^{\mathrm{sc}})$, by deck transformations. Show that this action agrees with the standard "conjugation" action of $\pi_1(X)$ on $\pi_n(X)$.

3. Show that an n-connected n-dimensional CW complex is contractible.

4. A space X is called *H-finite* if it is homotopic to a CW complex with finitely many cells, and a space Y is called *π-finite* if [it is a finite disjoint union of connected spaces, each of which satisfies] $\prod_{n=1}^{\infty} \pi_n(Y)$ is finite. Show that if X is H-finite and Y is π-finite, then $[X, Y]$ is finite.

5. Show that there is no retract $\mathbb{R}P^n \to \mathbb{R}P^k$ if $n > k > 0$.

6. Compute the action of $\pi_1 \mathbb{R}P^n$ on $\pi_n \mathbb{R}P^n$.

7. Recall that a space is called *acyclic* if its reduced homology vanishes. Show that if X is acyclic, then ΣX is contractible.

8. Show that a map between simply-connected CW complexes is a homotopy equivalence iff its mapping cylinder is contractible. Use the previous exercise to give a counterexample if the simply-connectedness hypothesis is dropped.

9. (a) Suppose that $f : X \to Y$ is a map of CW complexes, and let $f^{\mathrm{sc}} : X^{\mathrm{sc}} \to Y^{\mathrm{sc}}$ be the induced map. Show that f is a homotopy equivalence if $\pi_1(f)$ and $\mathrm{H}_\bullet(f^{\mathrm{sc}})$ are isos.

   (b) Conclude that if X and Y are n-dimensional CW complexes, with n-truncations $\tau_n X$ and $\tau_n Y$, then a map $f : X \to Y$ is a homotopy equivalence if $\tau_n f : \tau_n X \to \tau_n Y$ is.

10. (a) Let $C_p = \{1, t, \ldots, t^{p-1}\}$ denote the cyclic group of prime order p. Build a cell complex with p cells of each dimension, freely transitively permuted by C_p, as follows. Start with a single 0-cell e_0 and its images $te_0, t^2e_0, \ldots, t^{p-1}e_0$. Now attach a 2-cell e_1 with boundary $e_0 \sqcup te_0$. Now attach $p - 1$ more 2-cells $te_1, t^2e_1, \ldots,$ compatibly with the

C_p-action: $\partial t^i e_1 = t^i e_0 \sqcup t^{i+1} e_0$. Note that $\cup_i t^i e_1$ is an S^1, with basepoint e_0, and attach a 2-cell e_2 along it. Now attach $p-1$ more 2-cells $te_2, t^2 e_2, \dots$, again compatibly with the C_p action — only the basepoint changes. Now attach a 3-cell e_3 along $e_2 \cup te_2 \cong S^2$, and attach more 3-cells compatibly with the C_p-action. Note that you get an S^3. Fill that S^3 with a 4-cell e_4. Attach more 4-cells. Attach a 5-cell with boundary $e_4 \cup te_4$. Keep going.

Show that the resulting cell complex is contractible. Thus its quotient by the free C_p-action is a $K(C_p, 1)$.

Use this cell model of $K(C_p, 1)$ to compute $\mathrm{H}^\bullet(K(C_p, 1); A)$ for any A.

Compute $\dim \mathrm{H}^n(K(C_p^N, 1); \mathbb{F}_p)$.

(b) Show that C_p^2 cannot act freely on any sphere. Hint: If C_p^2 acts freely on S^n, then the quotient M is an n-dimensional closed manifold M. When $n > 1$ (the $n = 1$ case is easier), build a $K(C_p^2, 1)$ from M by attaching one cell of dimension $n+1$ and some cells of dimension $> n+1$. Conclude that $\dim \mathrm{H}^{n+1}(K(C_p^2, 1); \mathbb{F}_p) \leq 1$, which it ain't.

11. Suppose that $F \to E \to B$ is a fibre bundle such that $F \hookrightarrow E$ is homotopic to the constant map. Show that the LES in homotopy splits: $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. Conclude that $\pi_n \mathbb{R}P^n$, $\pi_{2n+1}\mathbb{C}P^n$, $\pi_{4n+3}\mathbb{H}P^n$, and $\pi_{15}\mathbb{O}P^1$ contain $\mathbb{Z}$-summands. Note that, in the $n = 1$ case, $\mathbb{K}P^1$ is a sphere of dimension $\dim_{\mathbb{R}} \mathbb{K}$.

12. Show that, up to homomotopy, there are precisely two maps $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$, the trivial one and a nontrivial one. Show that the nontrivial one induces the trivial map on $\widetilde{\mathrm{H}}_\bullet(-; \mathbb{Z})$ but a nontrivial map on $\widetilde{\mathrm{H}}^\bullet(-; \mathbb{Z})$. How is this consistent with the universal coefficient theorem? Hint: $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$.

    In fact, use the Bockstein for $\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times$ to show that the nontrivial map $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$ is simply the base-change along $\mathbb{R} \to \mathbb{C}$.

13. (a) Suppose that $E \to B \to C$ is a fibration. Show that the homotopy fibre of $E \to B$ is equivalent to ΩC. A fibration is called *principle* if, up to homotopy, it is of the form $\Omega C \to E \to B$ for some fibration $E \to B \to C$.

    (b) Show that a principle fibration $\Omega C \to E \xrightarrow{p} B$ is equivalent to a product $\Omega C \times B$ if and only if it has a section, i.e. a map $s : B \to E$ such that $ps = \mathrm{id}_B$. Hint for one direction: What is the homotopy fibre of the trivial map $B \to C$?