PhD Comprehensive Exam: Algebra Part II (nonspecialist) & Math 5055 Final Exam

Spring 2022

Thursday, 21 April 2022, 12-3PM

Your name:

Exam structure:

There are 9 questions on this exam. The pass mark is 60%.

- The PhD comprehensive exam consists of all 9 questions.
- The Math 5055 final exam consists of the final 6 questions.

Please indicate which exam you are taking.

1. Let G be a group.

   (a) Give the definition of *subgroup* of G.

   (b) Prove that if G is finite, then any nonempty subset of G which is closed under multiplication is a subgroup.

   (c) Give an example to show that the conclusion in part (b) can fail if G is infinite.

2. Let G be a finite group and p a prime. Suppose that $S, T \subset G$ are subgroups.

   (a) What does it mean for S to be a *p-subgroup*? What does it mean for S to be a *Sylow p-subgroup*?

   (b) Suppose that S is a Sylow p-subgroup and T is a p-subgroup. What can you say about the relationship between S and T?

   (c) Suppose that $G = S_6$ is the symmetric group on 6 elements. How many Sylow 3-subgroups are there?

   (d) Suppose that G has order $p^k \times m$ with $k \geq 1$ and $m < p$. Prove that G is not simple.

3. (a) How many (isomorphism classes of) abelian groups of order 300 are there? Justify your answer.

   (b) How many (isomorphism classes of) groups of order 10 are there? Justify your answer.

4. (a) Let E be a field, and let $G \subset \mathrm{Aut}(E)$ be a set of field automorphisms of E. What does it mean to say that an element of E is a G-*fixed point*?

   (b) Let $E^G \subset E$ denote the set of G-fixed points. Prove that $E^G \subset E$ is an extension of fields.

   (c) Give an example of an extension $F \subset E$ of fields such that $F \neq E^G$ for any set $G \subset \mathrm{Aut}(E)$ of field automorphisms.

5. Let $\theta = \sqrt{3 + \sqrt{11}}$.

   (a) Find the minimum polynomial f of θ over $\mathbb{Q}$.

   (b) Let K be the splitting field of f. Compute $\mathrm{Gal}(K/\mathbb{Q})$.

   (c) Find all intermediate subfields of $\mathbb{Q} \subset \mathbb{Q}(\theta)$.

   (d) Give an example of a transcendental extension of $\mathbb{Q}(\theta)$.

6. Let ζ_9 be a primitive 9th root of unity.

   (a) Find the minimum polynomial f of ζ_9 over $\mathbb{Q}$.

   (b) Prove that $\mathbb{Q} \subset \mathbb{Q}(\zeta_9)$ is Galois. What is its Galois group?

   (c) Find all intermediate subfields of $\mathbb{Q} \subset \mathbb{Q}(\zeta_9)$. Describe these fields as simple extensions over $\mathbb{Q}$, i.e. give a single generator for each intermediate extension.

7. Let F be a field of characteristic $p > 0$.

   (a) What is the *Frobenius endomorphism* of F?

   (b) Prove that the Frobenius endomorphism is an automorphism if and only if every finite extension $F \subset E$ is separable.

   (c) Why does this imply that every extension of finite fields is separable?

   (d) Prove that if F is a finite field, then $\mathrm{Aut}(F)$ is generated by the Frobenius endomorphism.

8. Find the Galois groups of the following polynomials over $\mathbb{Q}$ and over $\mathbb{R}$:

   (a) $x^3 - x^2 - 2x + 1$.
   Hint: The discriminant is 49.

   (b) $x^4 + 8x + 12$.
   Hint: The discriminant is $331776 = 576^2$ and the resolvent cubic is $x^3 - 48x - 64$.

9. (a) What does it mean for a finite group to be *solvable*? Why is the word "solvable" used for this concept? What is it that can be "solved"?

   (b) Let p be a prime. Prove that every finite p-group is solvable.

   (c) Give an example of an irreducible polynomial over $\mathbb{Q}$ of degree 5 whose Galois group is solvable. Give an example of an irreducible polynomial over $\mathbb{Q}$ of degree 5 whose Galois group is not solvable.