PhD Comprehensive Exam: Algebra Part II (nonspecialist) & Math 4055/5055 Final Exam

Spring 2022

Sample exam

Your name:

Exam structure:

There are 9 questions on this exam. The pass mark is 60%.

- The PhD comprehensive exam consists of all 9 questions.
- The Math 4055/5055 final exam consists of the final 6 questions.

Please indicate which exam you are taking.

1. Let G be a group.

   (a) What does it mean to say that a subgroup $K \subset G$ is *normal*?

   (b) Suppose that $H \subset G$ is a subgroup, and $K \subset G$ is a normal subgroup. Show that the *product*

   $$HK := \{hk | h \in H, k \in K\}$$

   is a subgroup of G.

2. Let G be a finite group.

(a) Define the *centre* $Z(G)$ of G and the *derived subgroup* $G' = [G, G]$ of G.

(b) Show that both $Z(G)$ and G' are normal subgroups of G.

(c) Let p be a prime. Show that if G is nonabelian of order p^3, then $Z(G) = G'$.

(d) Show that if G is nonabelian of order 6, then $Z(G) \neq G'$.

3. Prove that there is no simple group of order $980 = 2^2 \times 5 \times 7^2$. Hint: Constrain the number of Sylow subgroups.

4. (a) What does it mean for a field extension $F \subset E$ to have *degree* n?

   (b) Prove that if $F \subset E$ has degree $n < \infty$, then every element of E is a root of some polynomial over F of degree $\leq n$.

   (c) State, but do not prove, a relationship between the degree of $F \subset E$ and the order of $\mathrm{Gal}(E/F)$.

5. Consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{7})$.

   (a) Is this extension Galois?

   (b) Find all intermediate fields. Describe these fields as simple extensions over $\mathbb{Q}$, i.e. give a single generator for each intermediate extension.

   (c) Give an example of a transcendental extension of $\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{7})$.

6. Set $F = \mathbb{Q}(\sqrt{7})$, and set $K_1 = F(\sqrt{2+\sqrt{7}})$ and $K_2 = F(\sqrt{2-\sqrt{7}})$. Let $E = K_1K_2$ be the composite field.

   (a) Which of the following extensions are Galois?

$$\mathbb{Q} \subset F, \qquad \mathbb{Q} \subset K_1, \qquad \mathbb{Q} \subset K_2, \qquad \mathbb{Q} \subset E,$$
$$F \subset K_1, \qquad F \subset K_2, \qquad F \subset E, \qquad K_1 \subset E, \qquad K_2 \subset E$$

   (b) For the extensions in part (a) which are Galois, what is the Galois group?

7. Find the Galois groups of the following polynomials over $\mathbb{Q}$ and over $\mathbb{R}$:

   (a) $x^3 + 3x^2 + 2x - 1$.
   Hint: The discriminant is -23.

   (b) $x^4 - 4x^2 + x + 1$.
   Hint: The discriminant is 1957 and the resolvent cubic is $x^3 + 4x^2 - 4x + 15$.

8. (a) What does it mean for a field extension $F \subset E$ to be *separable*?

   (b) What does it mean for a field extension $F \subset E$ to be *purely inseparable*?

   (c) Give an example of a nontrivial field extension which is purely inseparable.

   (d) Give an example of a nontrivial field extension which is neither separable nor purely inseparable.

9. (a) Suppose that F is field. Prove that if $G \subset F^\times$ is a finite subgroup, then G is cyclic. Conclude that if F is finite, then $\mathbb{F}^\times$ is cyclic.

   (b) Describe the group $\mathbb{C}^\times$.

   (c) Prove that for each prime p and each positive integer n, there exists a field $\mathbb{F}_{p^n}$ of order p^n, and that it is unique up to isomorphism.