PhD Comprehensive Exam: Algebra Part II (nonspecialist) & Math 4055/5055 Final Exam

Spring 2024

April 17, 2024

Your name:

Exam structure:

There are 9 questions on this exam. The pass mark is 70%.

- The PhD comprehensive exam consists of any 8 of the 9 questions. You have three hours to complete the comprehensive exam.
- The Math 4055/5055 final exam consists of any 6 of the 9 questions. You have two hours to complete the final exam.

Please indicate which exam you are taking.

1. Normal subgroups.

   (a) Give the definition of *subgroup*. Give the definition of *normal subgroup*. Give an example of a normal subgroup. Give an example of a subgroup which is not normal.

   (b) Show that any subgroup of index 2 is normal.

2. The fundamental theorem of finite abelian groups.

   (a) List (up to isomorphism) all of the abelian groups of order 120. Explain/justify your answer.

   (b) List (up to isomorphism) all of the abelian groups of order 120 that are subgroups of the multiplicative group $F^\times$ of some field F. Explain/justify your answer.

3. Solvable groups.

   (a) When is a finite group *solvable*? Why is this name used?

   (b) When is a finite group *simple*? Why is this name used?

   (c) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible of prime degree $p \geq 5$, and suppose that $f(x)$ has exactly $p - 2$ real roots. Show that the roots of $f(x)$ cannot be expressed in terms of $+, -, \times, \div$, and $\sqrt[n]{-}$. You may use without proof that the alternating group A_p is simple, but you should explain how this is related to the problem.

4. Splitting fields.

(a) Give the definition of *degree* of a field extension. What is the degree of $\mathbb{Q} \subset \mathbb{Q}(\sqrt{7-\sqrt{2}})$? You do not need to justify your answer.

(b) Give the definition of when $\mathbb{Q} \subset K$ is a *splitting* field of $\sqrt{7-\sqrt{2}}$. Show that if K is a splitting field of $\sqrt{7-\sqrt{2}}$, then $K \ni \sqrt{47}$.

(c) What is the degree of a splitting field of $\sqrt{7-\sqrt{2}}$? You do not need to justify your answer.

(d) What is the automorphism group of the field $\mathbb{Q}(\sqrt{7-\sqrt{2}})$? You do not need to justify your answer.

(e) What is the automorphism group of a splitting field of $\sqrt{7-\sqrt{2}}$? You do not need to justify your answer.

5. Cyclotomic extensions and Galois correspondence.

   (a) Let ζ_{12} denote a primitive 12th root of unity. Show that $\mathbb{Q} \subset \mathbb{Q}(\zeta_{12})$ is Galois, and compute its Galois group. Also compute the minimal polynomial of ζ_{12}.

   (b) List all subfields of $\mathbb{Q}(\zeta_{12})$.

6. Computing Galois groups.

   (a) Compute the Galois group of $x^3 - 7x + 5$ over $\mathbb{Q}$ and over $\mathbb{R}$.
   Hint: the discriminant is 697.

   (b) Compute the Galois group of $x^4 + 3x^2 + 3x - 3$ over $\mathbb{Q}$.
   Hint: the resolvent cubic is $x^3 - 3x^2 + 12x - 45$ and the discriminant is -35991.

7. Finite fields.

   (a) Recall that $\mathbb{F}_{27}$ is generated, as a field, by a single element. How many elements of $\mathbb{F}_{27}$ are generators of $\mathbb{F}_{27}$ as a field?

   (b) Recall that $\mathbb{F}_{27}^{\times}$ is generated, as a group, by a single element. How many elements of $\mathbb{F}_{27}^{\times}$ are generators of $\mathbb{F}_{27}^{\times}$ as a group?

   (c) How many field automorphisms does $\mathbb{F}_{27}$ have? Into how many orbits does the set from question (7a) break under the action of $\mathrm{Aut}(\mathbb{F}_{27})$? What about the set from question (7b)?

   (d) Find the minimal polynomial of some element that generates $\mathbb{F}_{27}^{\times}$ as a group. (Hint: there is more than one answer.) Justify your answer.

8. The Frobenius map and inseparable extensions.

   (a) Let F be a field of positive characteristic. Define the *Frobenius endomorphism* $\mathrm{Frob}_F : F \to F$.

   (b) Give an example of a field F such that Frob_F is an automorphism.

   (c) Give an example of a field F such that Frob_F is a not an automorphism.

   (d) Give definitions of the following terms:

      - (in)separable polynomial
      - (in)separable extension
      - perfect field

   (e) State without proof the relationship between whether F is perfect and whether Frob_F is an automorphism.

9. Transcendental extensions.

   (a) What does it mean to say that a field extension $F \subset E$ is *transcendental*? Give an example of a transcendental extension.

   (b) Suppose that $F \subset E$ is a field extension. What does it mean that a subset $S \subset E$ is a *transcendence base* for E over F?

   (c) Show that any nontrivial field extension of $\mathbb{C}$ has uncountable dimension.