Math 4055/5055: Advanced Algebra II

Assignment 1

due February 1, 2024

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. (a) Show that $x^3 + 9x + 6$ is irreducible over $\mathbb{Q}$.

   Hint: It is enough to show that there are no roots in $\mathbb{Z}$ — why? Now check finitely many values.

   (b) Let θ be a root of x^3+9x+6, and $\mathbb{Q}[\theta]$ the corresponding field extension. In other words, $\mathbb{Q}[\theta] := \mathbb{Q}[x]/(x^3 + 9x + 6)$ with $\theta := x \mod x^3 + 9x + 6$. Compute $(1 + \theta)^{-1} \in \mathbb{Q}[\theta]$.

2. Show that $x^3 + x + 1$ is irreducible over $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. Let θ be a root, and compute its powers in $\mathbb{F}_2[\theta]$.

3. Let F be a field. In the field $F(x)$ of rational functions, let $u = x^3/(x + 1)$, and consider the subfield $F(u) \subset F(x)$. Compute the degree of this field extension.

   Hint: $F(x) = F(u)(x)$. Show that x is algebraic over $F(u)$, and find its minimal polynomial.

4. Let ℓ be a prime, $\mathbb{F}_\ell := \mathbb{Z}/\ell\mathbb{Z}$, and $F := \mathbb{F}_\ell(t)$ the field of rational functions. Show that $x^\ell - t$ is irreducible in $F[x]$.

   Hint: Suppose that $g(x)$ is a proper irreducible factor of $x^\ell - t$, and write $x^\ell - t = g(x)^a h(x)$ where g and h are coprime. Differentiate both sides and argue a contradiction unless $a = \ell$. Also use the irredicubility of t in $\mathbb{F}_\ell[t]$ to show that $\sqrt[\ell]{t} \notin F$.

5. Prove that the only unital ring endomorphism of $\mathbb{R}$ is the identity.

   Hint: $\mathbb{R}$ is totally ordered by: $x \leq y$ iff $y - x$ is a square.

6. A field F is *formally real* if -1 is not a sum of squares in F. Suppose that F is formally real and that $f(x) \in F[x]$ is irreducible of odd degree, and pick a root α of $f(x)$. Show that $F[\alpha]$ is formally real.

   Hint: Consider a counterexample of minimal degree. Show that there exists $g(x)$ of odd degree $< \deg(f)$ such that $-1 + f(x)g(x)$ is a sum of squares in $F[x]$. Show that $g(x)$ would give a new counterexample, violating the minimality of f.