Math 4055/5055: Advanced Algebra II

Assignment 2

due February 29, 2024

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. Let $F = \mathbb{Q}(\sqrt{-3})$. Let E denote the splitting field of $(x^3 - 2)(x^2 - 3)$ over F. Compute $\mathrm{Gal}(E/F)$. Compute the complete Galois correspondence for $F \subset E$: work out the poset of all subgroups of $\mathrm{Gal}(E/\mathbb{G})$ and use it to work out the poset of all subfields of E.

2. (a) Explain why the extensions $\mathbb{Q} \subset \mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\sqrt{1+\sqrt{5}})$ are each Galois, but that $\mathbb{Q} \subset \mathbb{Q}(\sqrt{1+\sqrt{5}})$ is not Galois.
   (b) Let E denote the splitting field of $(x^2 - 1)^2 - 5$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(E/\mathbb{Q})$. Compute the complete Galois correspondence for $\mathbb{Q} \subset E$: work out the poset of all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$ and use it to work out the poset of all subfields of E.

3. Let $p_1, \dots, p_n$ be a finite set of distinct primes. Let $E = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$.
   (a) Show that $\mathbb{Q} \subset E$ is Galois, and compute its Galois group. **Hint:** What is the degree $[E : \mathbb{Q}]$? Find that many distinct automorphisms of E. Why does this suffice?
   (b) How many subfields does E have? Describe/parameterize them.
   (c) Show that $E = \mathbb{Q}(\sqrt{p_1} + \cdots + \sqrt{p_n})$. **Hint:** Think about subfields.

4. (*Lagrange's Theorem of Natural Irrationalities*)
   Suppose given a diagram of field extensions

$$\begin{tikzcd} & & K \arrow[dr, hook] & & \\ F \arrow[r, hook] & K \cap L \arrow[ur, hook] \arrow[dr, hook] & & KL \arrow[r, hook] & E \\ & & L \arrow[ur, hook] & & \end{tikzcd}$$

   such that $F \subset K$ is finite and Galois. Prove that $L \subset KL$ is finite and Galois, and that $\mathrm{Gal}(KL/L) = \mathrm{Gal}(K/(K \cap L))$.

   Hints: $L \subset KL$ is the splitting field of some separable polynomial. (Why? So what?) Any F-linear automorphism of KL takes K to itself. (Why? So what?) Compute kernel and image of $\mathrm{Gal}(KL/L) \to \mathrm{Gal}(K/F)$.