

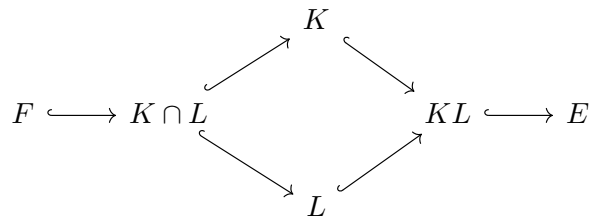
# Math 4055/5055: Advanced Algebra II

## Assignment 2

due February 29, 2024

Homework should be submitted as a single PDF attachment to `theo.jf@dal.ca`.

- Let  $F = \mathbb{Q}(\sqrt{-3})$ . Let  $E$  denote the splitting field of  $(x^3 - 2)(x^2 - 3)$  over  $F$ . Compute  $\text{Gal}(E/F)$ . Compute the complete Galois correspondence for  $F \subset E$ : work out the poset of all subgroups of  $\text{Gal}(E/F)$  and use it to work out the poset of all subfields of  $E$ .
- Explain why the extensions  $\mathbb{Q} \subset \mathbb{Q}(\sqrt{5})$  and  $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\sqrt{1 + \sqrt{5}})$  are each Galois, but that  $\mathbb{Q} \subset \mathbb{Q}(\sqrt{1 + \sqrt{5}})$  is not Galois.
  - Let  $E$  denote the splitting field of  $(x^2 - 1)^2 - 5$  over  $\mathbb{Q}$ . Compute  $\text{Gal}(E/\mathbb{Q})$ . Compute the complete Galois correspondence for  $\mathbb{Q} \subset E$ : work out the poset of all subgroups of  $\text{Gal}(E/\mathbb{Q})$  and use it to work out the poset of all subfields of  $E$ .
- Let  $p_1, \dots, p_n$  be a finite set of distinct primes. Let  $E = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n})$ .
  - Show that  $\mathbb{Q} \subset E$  is Galois, and compute its Galois group. **Hint:** What is the degree  $[E : \mathbb{Q}]$ ? Find that many distinct automorphisms of  $E$ . Why does this suffice?
  - How many subfields does  $E$  have? Describe/parameterize them.
  - Show that  $E = \mathbb{Q}(\sqrt{p_1} + \dots + \sqrt{p_n})$ . **Hint:** Think about subfields.
- (Lagrange's Theorem of Natural Irrationalities)  
Suppose given a diagram of field extensions



such that  $F \subset K$  is finite and Galois. Prove that  $L \subset KL$  is finite and Galois, and that  $\text{Gal}(KL/L) = \text{Gal}(K/(K \cap L))$ .

**Hints:**  $L \subset KL$  is the splitting field of some separable polynomial. (Why? So what?) Any  $F$ -linear automorphism of  $KL$  takes  $K$  to itself. (Why? So what?) Compute kernel and image of  $\text{Gal}(KL/L) \rightarrow \text{Gal}(K/F)$ .