# Math 4055/5055: Advanced Algebra II

Assignment 3

due February 29, 2024

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. (a) Suppose that $f(x) \in \mathbb{F}_3[x]$ is a monic irreducible cubic. Show that $f$ must divide $x^{27} - x$. Conversely, show that if $f$ is irreducible and divides $x^{27} - x$ then $f$ is either linear or cubic.

   (b) Use part (a) to (quickly!) count the number of monic irreducible cubics over $\mathbb{F}_3$.

   (c) List all the irreducible monic cubics over $\mathbb{F}_3$. **Hints:**

      i. Observe that, as functions $\mathbb{F}_3 \to \mathbb{F}_3$, the polynomial $x^3 - x$ always vanishes, whereas the polynomials 1 and $x^2 + 1$ never vanish. Use this to list at least four irreducible cubics over $\mathbb{F}_3$.

      ii. Observe that, as functions $\mathbb{F}_3 \to \mathbb{F}_3$, $x^2 - 1$ vanishes whenever $x \neq 0$, whereas $x^3$ and $x^3 + x$ vanish only when $x = 0$. Use this to list at least four irreducible cubics over $\mathbb{F}_3$.

2. (*Artin–Schreier extensions*) Let $p$ be a positive prime and $a \neq 0 \in \mathbb{F}_p$. Let $\mathbb{E} = \mathbb{F}_p[\alpha]$ where $\alpha$ is a root of $x^p - x - a$ over $\mathbb{F}_p$. Show that $\alpha \mapsto \alpha + 1$ extends to an automorphism of $\mathbb{E}$. Conclude that $x^p - x - a$ is irreducible and that $\mathbb{E}$ is its splitting field. How does $\alpha \mapsto \alpha + 1$ relate to the Frobenius endomorphism of $\mathbb{E}$?

3. Recall that a field $F$ is *perfect* if all of its algebraic extensions are separable. Equivalently, a field $F$ is perfect if it has no irreducible inseparable polynomials. Recall also that all fields of characteristic zero are perfect. So let's assume that $F$ has characteristic $p > 0$. Recall also that every field of characteristic $p$ has a *Frobenius endomorphism* $\phi_F : F \to F$, $x \mapsto x^p$.

   (a) Suppose that the Frobenius endomorphism $\phi_F$ of $F$ is not an automorphism. Give an example of an inseparable extension $F \subset E$ of degree $p$.

   (b) Conversely, suppose that $F$ admits a finite-degree inseparable extension $F \subset E$, and let $q(x) \in F[x]$ be the minimal polynomial of some inseparable element in $E \smallsetminus F$. Remind why $q(x) \in F[x^p]$. The ring $F[x]$ has a Frobenius endomorphism $\phi_{F[x]}$. Why does it land in $F[x^p]$? Show that $\phi_F : F \to F$ is an automorphism if and only if $\phi_{F[x]} : F[x] \to F[x^p]$ is an isomorphism. Conclude that if $q(x) \in F[x^p]$ is irreducible in $F[x]$, then $\phi_F$ could not be an automorphism.

   In summary, you have shown that a field $F$ of characteristic $p$ is perfect if and only if $\phi_F$ is an automorphism. Use this to give a proof that every finite field is perfect which does not require any classification of finite fields.