Math 4055/5055: Advanced Algebra II

Assignment 4

due March 21, 2024

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`.

1. List (up to isomorphism) all groups of order 75:
   (a) Use Sylow theory to show that if G is a group of order 75, then its Sylow 5-subgroup is normal. Conclude that $G = P_5 \rtimes P_3$ is a semidirect product, where P_5 and P_3 are the Sylow 5- and 3-subgroups respectively.
   (b) List all the groups of order 25. Calculate the orders of their automorphism groups.
   (c) For each group A of order 25, use Sylow theory to calculate all possible conjugacy classes of action of C_3 on A. Hint: For one of them, there should be only the trivial action, and for the other, there should be a unique nontrivial action.
2. Determine the Galois groups of the following polynomials over the fields indicated:
   (a) $x^4 - 5$ over $\mathbb{Q}$; over $\mathbb{Q}[\sqrt{-1}]$; over $\mathbb{Q}[\sqrt{5}]$; over $\mathbb{Q}[\sqrt{-5}]$.
   (b) $x^3 - x - 1$ over $\mathbb{Q}$; over $\mathbb{Q}[\sqrt{-23}]$.
   (c) $x^4 + 3x^2 + 3x - 2$ over $\mathbb{Q}$; over $\mathbb{R}$.
   (d) $x^5 - 6x + 3$ over $\mathbb{Q}$.
3. Let $F \to E$ be a Galois extension, with Galois group G. The *trace* is the function $\mathrm{tr} : E \to F$ defined by
$$\mathrm{tr}(\alpha) = \sum_{g \in G} g\alpha,$$
   and the *norm* is function $\mathrm{norm} : E \to F$ defined by
$$\mathrm{norm}(\alpha) = \prod_{g \in G} g\alpha.$$
   (a) What are the trace and norm for the extension $\mathbb{R} \to \mathbb{C}$?
   (b) Think of E as an F-vector space. Describe $\mathrm{tr}(\alpha)$ and $\mathrm{norm}(\alpha)$ in terms of the F-linear map $\alpha \cdot (-) : E \to E$. In particular, explain the name "trace."
   (c) Let $n = [E : F] = \#G$. Show that if $\alpha \in E$, then α is the root of a degree-n polynomial $f(x) \in F[x]$ of the form
$$f(x) = x^n - \mathrm{tr}(\alpha)x^{n-1} + \cdots \pm \mathrm{norm}(\alpha)$$
   where the sign is $\pm = (-1)^n$. What are the dotted-out coefficients?
   (d) Show that if F, E are finite fields, then both $\mathrm{tr} : E \to F$ and $\mathrm{norm} : E \to F$ are surjective.