Math 4055/5055: Advanced Algebra II

Assignment 5

optional

Commutative inseparability

Fix a positive prime p, and assume throughout this section that all fields under consideration have characteristic p.

1. Show that if $F \subset E$ is a finite-degree field extension such that $[E : F]$ is not divisible by p, then $F \subset E$ is separable.

2. Let $F \subset E$ be an algebraic extension. Suppose that $\alpha \neq 0 \in E$ is separable over F and $\beta \neq 0 \in E$ is purely inseparable over F. Prove that $F[\alpha, \beta] = F[\alpha + \beta] = F[\alpha\beta]$.

3. Recall that the *separable degree* $[E : F]_s$ of a field extension is the degree $[E_s : F]$ where $E_s \subset E$ is the subfield of separable (over F) elements. Suppose that $E = F[\alpha]$ is a simple algebraic extension. What is $[E : F]_\alpha$ in terms of (the minimal polynomial of) α?

4. Fix a finite-degree extension $F \subset E$. Let $\bar{F} = \bar{E}$ denote the algebraic closure of F (why is $\bar{F} = \bar{E}$?), and let $F^s = \bar{F}_s \subset \bar{F}$ denote the separable closure. Show that the sets $\hom_F(E, \bar{F})$ and $\hom_F(E_s, F^s)$ are canonically isomorphic (where the hom is in the category of fields over F). Conclude that $\hom(E, \bar{F})$ is of cardinality equal to $[E : F]_s$. Conclude that $[:]_s$ is multiplicative.

Noncommutative separability

5. Let F be a field and A an associative and unital, but possibly noncommutative, F-algebra. Show that the multiplication map $m : A \otimes A \to A$ is always a homomorphism of A-bimodules. Show that $m : A \otimes A \to A$ is a homomorphism of algebras if and only if A is commutative.

6. A is called *separable* if there exists an A-bilinear splitting $\Delta : A \to A \otimes A$ of the multiplication map m. Such a Δ is called a *separation* of A. Show that Δ is determined by the element $u = \Delta(1) = \sum_i u_i^{(1)} \otimes u_i^{(2)} \in A \otimes A$, and that u must satisfy the equations
$$\sum_i a u_i^{(1)} \otimes u_i^{(2)} = \sum_i u_i^{(1)} \otimes u_i^{(2)} a \in A \otimes A, \qquad \sum_i u_i^{(1)} u_i^{(2)} = 1 \in A$$
and that these are the only conditions.

7. Suppose that $F \subset E = A$ is finite-degree a separable field extension. Then $E = F[\alpha]$ for some $\alpha \in E$, and let $f(x) \in F[x]$ denote the minimal polynomial of α, and $f'(x)$ its derivative. In $E[x]$, we can factor $f(x) = (x - \alpha)\sum_n b_n x^n$, and by separability, $f'(\alpha) \neq 0$. Show that
$$u = \sum \alpha^n \otimes \frac{b_n}{f'(\alpha)} \in E \otimes E$$

provides a separation for E.

8. Show that every separation $\Delta : A \to A \otimes A$ is *coassociative*:

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta : A \to A \otimes A \otimes A.$$

   The name is because the dual map $\Delta^* : A^* \otimes A^* \to (A \otimes A)^* \to A$ is associative. Hint: Precompose both sides with $\mathrm{id}_A = m \circ \Delta$.

9. Recall that a finite dimensional algebra A is *semisimple* if all of its finite-dimensional left modules are projective. Show that if A is finite-dimensional and separable, then it is semisimple. Hint: Let M be a left A-module. Use the isomorphism $A \otimes_A M \cong M$. Now use Δ to write A as a direct summand of $A \otimes A$.

10. Prove that the tensor product of separable algebras is again separable.

11. Suppose that A is finite-dimensional, and let A^{op} denote its opposite algebra. The *enveloping algebra* of A is $A^e := A \otimes A^{\mathrm{op}}$. Prove that A is separable if and only if A^e is semisimple. Hint: A^e-modules are the same as A-bimodules. In one direction, use semisimplicity of A^e to find a separation of A. In the other direction, use the previous two questions.

12. Let F be an imperfect field, and $E = F[\sqrt[p]{a}]$ a degree-p inseparable extension (i.e. $a \in F$ is some element which does not have a pth root in F). Show that $E^e \cong E \otimes E \cong E[\epsilon]/(\epsilon^p)$ and that under this isomorphism, m becomes the quotient map $\epsilon \mapsto 0$. Show that this map does not split E^e-linearly. Explain why this shows that the converse of question 9 fails.

13. More generally, show that if $F \subset E$ is any inseparable extension of fields, then $E \otimes_F E$ contains a nilpotent element, and that this prevents E from being separable in the sense of question 6.

14. Compute (the dimension of) the space of separations for the quaternion algebra $A = \mathbb{H}$ thought of as an algebra over the real field $F = \mathbb{R}$. Hint: $\mathbb{H}^e \cong \mathrm{Mat}_4(\mathbb{R})$. This algebra is semisimple with a unique simple left module. Who are $A \otimes A$ and A in terms of that unique simple left module? Hint: The category of left $\mathrm{Mat}_4(\mathbb{R})$-modules is equivalent to $\mathrm{Vec}_{\mathbb{R}}$ via the functor that sends the unique simple left module to $\mathbb{R}^1$.