Problem Set 1: Closed Linear Groups

1. (a) Show that the orthogonal groups $\mathrm{O}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{C})$ have two connected components, the identity component being the special orthogonal group SO_n, and the other consisting of orthogonal matrices of determinant -1.

   (b) Show that the center of $\mathrm{O}(n)$ is $\{\pm I_n\}$.

   (c) Show that if n is odd, then $\mathrm{SO}(n)$ has trivial center and $\mathrm{O}(n) \cong \mathrm{SO}(n) \times (\mathbb{Z}/2\mathbb{Z})$ as a Lie group.

   (d) Show that if n is even, then the center of $\mathrm{SO}(n)$ has two elements, and $\mathrm{O}(n)$ is a semidirect product $\mathrm{SO}(n) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{SO}(n)$ by a non-trivial outer automorphism of order 2.

2. Construct a smooth group homomorphism $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ which induces an isomorphism of Lie algebras and identifies $\mathrm{SO}(3)$ with the quotient of $\mathrm{SU}(2)$ by its center $\{\pm I\}$.

3. Construct an isomorphism of $\mathrm{GL}(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $\mathrm{SL}(n+1, \mathbb{C})$.

4. Show that the map $\mathbb{C}^* \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

5. Find the Lie algebra of the group $U \subseteq \mathrm{GL}(n, \mathbb{C})$ of upper-triangular matrices with 1 on the diagonal. Show that for this group, the exponential map is a diffeomorphism of the Lie algebra onto the group.

6. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_\mathbb{R}$ such that that $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \oplus i\mathfrak{g}_\mathbb{R}$, or equivalently, such that the canonical map $\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \to \mathfrak{g}$ given by scalar multiplication is an isomorphism. A real form of a (connected) complex closed linear group G is a (connected) closed real subgroup $G_\mathbb{R}$ such that $\mathrm{Lie}(G_\mathbb{R})$ is a real form of $\mathrm{Lie}(G)$.

   (a) Show that $\mathrm{U}(n)$ is a compact real form of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ is a compact real form of $\mathrm{SL}(n, \mathbb{C})$.

   (b) Show that $\mathrm{SO}(n, \mathbb{R})$ is a compact real form of $\mathrm{SO}(n, \mathbb{C})$.

   (c) Show that $\mathrm{Sp}(n, \mathbb{R})$ is a compact real form of $\mathrm{Sp}(n, \mathbb{C})$.