Problem Set 2: Differential Geometry

1. (a) Show that the composition of two immersions is an immersion.

   (b) Show that an immersed submanifold $N \subseteq M$ is always a closed submanifold of an open submanifold, but not necessarily an open submanifold of a closed submanifold.

2. Prove that if $f : N \to M$ is a smooth map, then $(\mathrm{d}f)_p$ is surjective if and only if there are open neighborhoods U of p and V of $f(p)$, and an isomorphism $\psi : V \times W \to U$, such that $f \circ \psi$ is the projection on V.

   In particular, deduce that the fibers of f meet a neighborhood of p in immersed closed submanifolds of that neighborhood.

3. Prove the implicit function theorem: a map (of sets) $f : M \to N$ between manifolds is smooth if and only if its graph is an immersed closed submanifold of $M \times N$.

4. Prove that the curve $y^2 = x^3$ in $\mathbb{R}^2$ is not an immersed submanifold.

5. Let M be a complex holomorphic manifold, p a point of M, X a holomorphic vector field. Show that X has a complex integral curve γ defined on an open neighborhood U of 0 in $\mathbb{C}$, and unique on U if U is connected, which satisfies the usual defining equation but in a complex instead of a real variable t.

   Show that the restriction of γ to $U \cap \mathbb{R}$ is a real integral curve of X, when M is regarded as a real analytic manifold.

6. Let $\mathrm{SL}(2, \mathbb{C})$ act on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = (az + b)/(cz + d)$. Determine explicitly the vector fields $f(z)\partial_z$ corresponding to the infinitesimal action of the basis elements

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

   of $\mathfrak{sl}(2, \mathbb{C})$, and check that you have constructed a Lie algebra homomorphism by computing the commutators of these vector fields.

7. (a) Describe the map $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = \mathrm{Mat}(n, \mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n)$ given by the infinitesimal action of $\mathrm{GL}(n, \mathbb{R})$.

   (b) Show that $\mathfrak{so}(n, \mathbb{R})$ is equal to the subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ consisting of elements whose infinitesimal action is a vector field tangential to the unit sphere in $\mathbb{R}^n$.

8. (a) Let X be an analytic vector field on M all of whose integral curves are unbounded (i.e., they are defined on all of $\mathbb{R}$). Show that there exists an analytic action of $R = (\mathbb{R}, +)$ on M such that X is the infinitesimal action of the generator ∂_t of $\mathrm{Lie}(\mathbb{R})$.

   (b) More generally, prove the corresponding result for a family of n commuting vector fields X_i and action of $\mathbb{R}^n$.

9. (a) Show that the matrix $\begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}$ belongs to the identity component of $\mathrm{GL}(2, \mathbb{R})$ for all positive real numbers a, b.

   (b) Prove that if $a \neq b$, the above matrix is not in the image $\exp(\mathfrak{gl}(2, \mathbb{R}))$ of the exponential map.