Problem Set 3: General Theory of Lie Groups

1. (a) Let S be a commutative $\mathbb{K}$-algebra. Show that a linear operator $d : S \to S$ is a derivation if and only if it annihilates 1 and its commutator with the operator of multiplication by every function is the operator of multiplication by another function.

   (b) Grothendieck's inductive definition of differential operators on S goes as follows: the differential operators of order zero are the operators of multiplication by functions; the space $D_{\leq n}$ of operators of order at most n is then defined inductively for $n > 0$ by $D_{\leq n} = \{d \text{ s.t. } [d, f] \in D_{\leq n-1} \text{ for all } f \in S\}$. Show that the differential operators of all orders form a filtered algebra D, and that when S is the algebra of smooth functions on an open set in $\mathbb{R}^n$ [or $\mathbb{C}^n$], D is a free left S-module with basis consisting of all monomials in the coordinate derivations $\partial/\partial x^i$.

2. Calculate all terms of degree ≤ 4 in the Baker–Campbell–Hausdorff formula.

3. Let $F(d)$ be the free Lie algebra on generators $x_1, \ldots, x_d$. It has a natural $\mathbb{N}^d$ grading in which $F(d)_{(k_1,\ldots,k_d)}$ is spanned by bracket monomials containing k_i occurences of each generator X_i. Use the PBW theorem to prove the generating function identity

$$\prod_{\mathbf{k}} \frac{1}{(1 - t_1^{k_1} \ldots t_d^{k_d})^{\dim F(d)_{(k_1,\ldots,k_d)}}} = \frac{1}{1 - (t_1 + \cdots + t_d)}$$

4. Words in the symbols $x_1, \ldots, x_d$ form a monoid under concatentation, with identity the empty word. Define a *primitive word* to be a non-empty word that is not a power of a shorter word. A *primitive necklace* is an equivalence class of primitive words under rotation. Use the generating function identity in Problem 3 to prove that the dimension of $F(d)_{k_1,\ldots,k_d}$ is equal to the number of primitive necklaces in which each symbol x_i appears k_i times.

5. A *Lyndon word* is a primitive word that is the lexicographically least representative of its primitive necklace.

   (a) Prove that w is a Lyndon word if and only if w is lexicographically less than v for every factorization $w = uv$ such that u and v are non-empty.

   (b) Prove that if $w = uv$ is a Lyndon word of length > 1 and v is the longest proper right factor of w which is itself a Lyndon word, then u is also a Lyndon word. This factorization of w is called its *right standard factorization.*

   (c) To each Lyndon word w in symbols $x_1, \ldots, x_d$ associate the bracket polynomial $p_w = x_i$ if $w = x_i$ has length 1, or, inductively, $p_w = [p_u, p_v]$, where $w = uv$ is the right standard factorization, if w has length > 1. Prove that the elements p_w form a basis of $F(d)$.

6. Prove that if q is a power of a prime, then the dimension of the subspace of total degree $k_1 + \cdots + k_q = n$ in $F(q)$ is equal to the number of monic irreducible polynomials of degree n over the field with q elements.

7. This problem outlines an alternative proof of the PBW theorem.

(a) Let $L(d)$ denote the Lie subalgebra of $\mathcal{T}(x_1, \ldots, x_d)$ generated by $x_1, \ldots, x_d$. Without using the PBW theorem—in particular, without using $F(d) = L(d)$—show that the value given for $\dim F(d)_{(k_1,\ldots,k_d)}$ by the generating function in Problem 3 is a lower bound for $\dim L(d)_{(k_1,\ldots,k_d)}$.

(b) Show directly that the Lyndon monomials in Problem 5(b) span $F(d)$.

(c) Deduce from (a) and (b) that $F(d) = L(d)$ and that the PBW theorem holds for $F(d)$.

(d) Show that the PBW theorem for a Lie algebra $\mathfrak{g}$ implies the PBW theorem for $\mathfrak{g}/\mathfrak{a}$, where $\mathfrak{a}$ is a Lie ideal, and so deduce PBW for all finitely generate Lie algebras from (c).

(e) Show that the PBW theorem for arbitrarty Lie algebras reduces to the finitely generated case.

8. Let $b(x, y)$ be the Baker–Campbell–Hausdorff series, i.e., $e^{b(x,y)} = e^x e^y$ in noncommuting variables x, y. Let $F(x, y)$ be its linear term in y, that is, $b(x, sy) = x + sF(x, y) + O(s^2)$.

(a) Show that $F(x, y)$ is characterized by the identity

$$\sum_{k,l\geq 0} \frac{x^k\, F(x, y)\, x^l}{(k + l + 1)!} = e^x y.$$

(b) Let λ, ρ denote the operators of left and right multiplication by x, and let f be the series in two commuting variables such that $F(x, y) = f(\lambda, \rho)(y)$. Show that

$$f(\lambda, \rho) = \frac{\lambda - \rho}{1 - e^{\rho-\lambda}}$$

(c) Deduce that

$$F(x, y) = \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}}(y).$$

9. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $0 \in U' \subseteq U \subseteq \mathfrak{g}$ and $e \in V' \subseteq V \subseteq G$ open neighborhoods such that exp is an isomorphism of U onto V, $\exp(U') = V'$, and $V'V' \subseteq V$. Define $\beta : U' \times U' \to U$ by $\beta(x, y) = \log(\exp(x) \exp(y))$, where $\log : V \to U$ is the inverse of exp.

(a) Show that $\beta(x, (s + t)y) = \beta(\beta(x, ty), sy)$ whenever all arguments are in U'.

(b) Show that the series $(\operatorname{ad} x)/(1 - e^{-\operatorname{ad} x})$, regarded as a formal power series in the coordinates of x with coefficients in the space of linear endomorphisms of $\mathfrak{g}$, converges for all x in a neighborhood of 0 in $\mathfrak{g}$.

(c) Show that on some neighborhood of 0 in $\mathfrak{g}$, $\beta(x, ty)$ is the solution of the initial value problem

$$\beta(x, 0) = x \tag{0.0.0.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(x, ty) = F(\beta(x, ty), y), \tag{0.0.0.2}$$

where $F(x, y) = \left((\operatorname{ad} x)/(1 - e^{-\operatorname{ad} x})\right)(y)$.

(d) Show that the Baker–Campbell–Hausdorff series $b(x, y)$ also satisfies the identity in part (a), as an identity of formal power series, and deduce that it is the formal power series solution to the IVP in part (c), when $F(x, y)$ is regarded as a formal series.

(e) Deduce from the above an alternative proof that $b(x, y)$ is given as the sum of a series of Lie bracket polynomials in x and y, and that it converges to $\beta(x, y)$ when evaluated on a suitable neighborhood of 0 in $\mathfrak{g}$.

(f) Use part (c) to calculate explicitly the terms of $b(x, y)$ of degree 2 in y.

10. (a) Show that the Lie algebra $\mathfrak{so}(3, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

    (b) Construct a Lie group homomorphism $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

11. (a) Show that the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$.

    (b) Construct a Lie group homomorphism $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(4, \mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

12. Show that every closed subgroup H of a Lie group G is a Lie subgroup, so that the inclusion $H \hookrightarrow G$ is a closed immersion.

13. Let G be a Lie group and H a closed subgroup. Show that G/H has a unique manifold structure such that the action of G on it is smooth.

14. Show that the intersection of two Lie subgroups H_1, H_2 of a Lie group G can be given a canonical structure of Lie subgroup so that its Lie algebra is $\mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2) \subseteq \mathrm{Lie}(G)$.

15. Find the dimension of the closed linear group $\mathrm{SO}(p, q, \mathbb{R}) \subseteq \mathrm{SL}(p+q, \mathbb{R})$ consisting of elements which preserve a non-degenerate symmetric bilinear form on $\mathbb{R}^{p+q}$ of signature (p, q). When is this group connected?

16. Show that the kernel of a Lie group homomorphism $G \to H$ is a closed subgroup of G whose Lie algebra is equal to the kernel of the induced map $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$.

17. Show that if H is a normal Lie subgroup of G, then $\mathrm{Lie}(H)$ is a Lie ideal in $\mathrm{Lie}(G)$.