Problem Set 4: General Theory of Lie Algebras

1. Classify the 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field $\mathbb{K}$ of characteristic zero by showing that if $\mathfrak{g}$ is not a direct product of smaller Lie algebras, then either

   - $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{K})$,
   - $\mathfrak{g}$ is isomorphic to the nilpotent Heisenberg Lie algebra $\mathfrak{h}$ with basis X, Y, Z such that Z is central and $[X, Y] = Z$, or
   - $\mathfrak{g}$ is isomorphic to a solvable algebra $\mathfrak{s}$ which is the semidirect product of the abelian algebra $\mathbb{K}^2$ by an invertible derivation. In particular $\mathfrak{s}$ has basis X, Y, Z such that $[Y, Z] = 0$, and $\operatorname{ad} X$ acts on $\mathbb{K}Y + \mathbb{K}Z$ by an invertible matrix, which, up to change of basis in $\mathbb{K}Y + \mathbb{K}Z$ and rescaling X, can be taken to be either $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some nonzero $\lambda \in \mathbb{K}$.

2. (a) Show that the Heisenberg Lie algebra $\mathfrak{h}$ in Problem 1 has the property that Z acts nilpotently in every finite-dimensional module, and as zero in every simple finite-dimensional module.

   (b) Construct a simple infinite-dimensional $\mathfrak{h}$-module in which Z acts as a non-zero scalar. [Hint: take X and Y to be the operators $\frac{\mathrm{d}}{\mathrm{d}t}$ and t on $\mathbb{K}[t]$.]

3. Construct a simple 2-dimensional module for the Heisenberg algebra $\mathfrak{h}$ over any field $\mathbb{K}$ of characteristic 2. In particular, if $\mathbb{K} = \bar{\mathbb{K}}$, this gives a counterexample to Lie's theorem in non-zero characteristic.

4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$.

   (a) Show that the intersection $\mathfrak{n}$ of the kernels of all finite-dimensional simple $\mathfrak{g}$-modules can be characterized as the largest ideal of $\mathfrak{g}$ which acts nilpotently in every finite-dimensional $\mathfrak{g}$-module. It is called the *nilradical* of $\mathfrak{g}$.

   (b) Show that the nilradical of $\mathfrak{g}$ is contained in $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$.

   (c) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra and V a $\mathfrak{g}$-module. Given a linear functional $\lambda : \mathfrak{h} \to \mathbb{K}$, define the associated weight space to be $V_\lambda = \{v \in V : Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$. Assuming $\operatorname{char}(\mathbb{K}) = 0$, adapt the proof of Lie's theorem to show that if $\mathfrak{h}$ is an ideal and V is finite-dimensional, then V_λ is a $\mathfrak{g}$-submodule of V.

   (d) Show that if $\operatorname{char}(\mathbb{K}) = 0$ then the nilradical of $\mathfrak{g}$ is equal to $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$. [Hint: assume without loss of generality that $\mathbb{K} = \bar{\mathbb{K}}$ and obtain from Lie's theorem that any finite-dimensional simple $\mathfrak{g}$-module V has a non-zero weight space for some weight λ on $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$. Then use (c) to deduce that $\lambda = 0$ if V is simple.]

5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$, $\operatorname{char}(\mathbb{K}) = 0$. Prove that the largest nilpotent ideal of $\mathfrak{g}$ is equal to the set of elements of $\mathfrak{r} = \operatorname{rad} \mathfrak{g}$ which act nilpotently in the adjoint action on $\mathfrak{g}$, or equivalently on $\mathfrak{r}$. In particular, it is equal to the largest nilpotent ideal of $\mathfrak{r}$.

6. Prove that the Lie algebra $\mathfrak{sl}(2, \mathbb{K})$ of 2×2 matrices with trace zero is simple, over a field $\mathbb{K}$ of any characteristic $\neq 2$. In characteristic 2, show that it is nilpotent.

7. In this exercise, we'll deduce from the standard functorial properties of Ext groups and their associated long exact sequences that $\operatorname{Ext}^1(N, M)$ bijectively classifies extensions $0 \to M \to V \to N \to 0$ up to isomorphism, for modules over any associative ring with unity.

   (a) Let F be a free module with a surjective homomorphism onto N, so we have an exact sequence $0 \to K \to F \to N \to 0$. Use the long exact sequence to produce an isomorphism of $\operatorname{Ext}^1(N, M)$ with the cokernel of $\operatorname{Hom}(F, M) \to \operatorname{Hom}(K, M)$.

   (b) Given $\phi \in \operatorname{Hom}(K, M)$, construct V as the quotient of $F \oplus M$ by the graph of $-\phi$ (note that this graph is a submodule of $K \oplus M \subseteq F \oplus M$).

   (c) Use the functoriality of Ext and the long exact sequences to show that the characteristic class in $\operatorname{Ext}^1(N, M)$ of the extension constructed in (b) is the element represented by the chosen ϕ, and conversely, that if ϕ represents the characteristic class of a given extension, then the extension constructed in (b) is isomorphic to the given one.

8. Calculate $\operatorname{Ext}^i(\mathbb{K}, \mathbb{K})$ for all i for the trivial representation $\mathbb{K}$ of $\mathfrak{sl}(2, \mathbb{K})$, where $\operatorname{char}(\mathbb{K}) = 0$. Conclude that the theorem that $\operatorname{Ext}^i(M, N) = 0$ for $i = 1, 2$ and all finite-dimensional representations M, N of a semi-simple Lie algebra $\mathfrak{g}$ does not extend to $i > 2$.

9. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that $\operatorname{Ext}^1(\mathbb{K}, \mathbb{K})$ can be canonically identified with the dual space of $\mathfrak{g}/\mathfrak{g}'$, and therefore also with the set of 1-dimensional $\mathfrak{g}$-modules, up to isomorphism.

10. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that $\operatorname{Ext}^1(\mathbb{K}, \mathfrak{g})$ can be canonically identified with the quotient $\operatorname{Der}(\mathfrak{g})/\operatorname{Inn}(\mathfrak{g})$, where $\operatorname{Der}(\mathfrak{g})$ is the space of derivations of $\mathfrak{g}$, and $\operatorname{Inn}(\mathfrak{g})$ is the subspace of inner derivations, that is, those of the form $d(x) = [y, x]$ for some $y \in \mathfrak{g}$. Show that this also classifies Lie algebra extensions $\hat{\mathfrak{g}}$ containing $\mathfrak{g}$ as an ideal of codimension 1.

11. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$, $\operatorname{char}(\mathbb{K}) = 0$. The Malcev-Harish-Chandra theorem says that all Levi subalgebras $\mathfrak{s} \subseteq \mathfrak{g}$ are conjugate under the action of the group $\exp \operatorname{ad} \mathfrak{n}$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$ (note that $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}$, so the power series expression for $\exp \operatorname{ad} X$ reduces to a finite sum when $X \in \mathfrak{n}$).

    (a) Show that the reduction we used to prove Levi's theorem by induction in the case where the radical $\mathfrak{r} = \operatorname{rad} \mathfrak{g}$ is not a minimal ideal also works for the Malcev-Harish-Chandra theorem. More precisely, show that if $\mathfrak{r}$ is nilpotent, the reduction can be done using any nonzero ideal $\mathfrak{m}$ properly contained in $\mathfrak{r}$. If $\mathfrak{r}$ is not nilpotent, use Problem 4 to show that $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, then make the reduction by taking $\mathfrak{m}$ to contain $[\mathfrak{g}, \mathfrak{r}]$.

    (b) In general, given a semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is an abelian ideal, show that $\operatorname{Ext}^1_{\mathcal{U}(\mathfrak{h})}(\mathbb{K}, \mathfrak{m})$ classifies subalgebras complementary to $\mathfrak{m}$, up to conjugacy by the action of $\exp \operatorname{ad} \mathfrak{m}$. Then use the vanishing of $\operatorname{Ext}^1(M, N)$ for finite-dimensional modules over a semi-simple Lie algebra to complete the proof of the Malcev-Harish-Chandra theorem.