Problem Set 5: Classification of Semisimple Lie Algebras

1. (a) Show that $\mathrm{SL}(2,\mathbb{R})$ is topologically the product of a circle and two copies of $\mathbb{R}$, hence it is not simply connected.

   (b) Let S be the simply connected cover of $\mathrm{SL}(2,\mathbb{R})$. Show that its finite-dimensional complex representations, i.e., real Lie group homomorphisms $S \to \mathrm{GL}(n,\mathbb{C})$, are determined by corresponding complex representations of the Lie algebra $\mathrm{Lie}(S)^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$, and hence factor through $\mathrm{SL}(2,\mathbb{R})$. Thus S is a simply connected real Lie group with no faithful finite-dimensional representation.

2. (a) Let U be the group of 3×3 upper-unitriangular complex matrices. Let $\Gamma \subseteq U$ be the cyclic subgroup of matrices
$$\begin{bmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
   where $m \in \mathbb{Z}$. Show that $G = U/\Gamma$ is a (non-simply-connected) complex Lie group that has no faithful finite-dimensional representation.

   (b) Adapt the solution to Set 4, Problem 2(b) to construct a faithful, irreducible infinite-dimensional linear representation V of G.

3. Following the outline below, prove that if $\mathfrak{h} \subseteq \mathfrak{gl}(n,\mathbb{C})$ is a real Lie subalgebra with the property that every $X \in \mathfrak{h}$ is diagonalizable and has purely imaginary eigenvalues, then the corresponding connected Lie subgroup $H \subseteq GL(n,\mathbb{C})$ has compact closure (this completes the solution to Set 1, Problem 7).

   (a) Show that $\operatorname{ad} X$ is diagonalizable with imaginary eigenvalues for every $X \in \mathfrak{h}$.

   (b) Show that the Killing form of $\mathfrak{h}$ is negative semidefinite and its radical is the center of $\mathfrak{h}$. Deduce that $\mathfrak{h}$ is reductive and the Killing form of its semi-simple part is negative definite. Hence the Lie subgroup corresponding to the semi-simple part is compact.

   (c) Show that the Lie subgroup corresponding to the center of $\mathfrak{h}$ is a dense subgroup of a compact torus. Deduce that the closure of H is compact.

   (d) Show that H is compact — that is, closed — if and only if it further holds that the center of $\mathfrak{h}$ is spanned by matrices whose eigenvalues are rational multiples of i.

4. Let $V_n = \mathcal{S}^n(\mathbb{C}^2)$ be the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

   (a) Show that for $m \le n$, $V_m \otimes V_n \cong V_{n-m} \oplus V_{n-m+2} \oplus \cdots \oplus V_{n+m}$, and deduce that the decomposition into irreducibles is unique.

   (b) Show that in any decomposition of $V_1^{\otimes n}$ into irreducibles, the multiplicity of V_n is equal to 1, the multiplicity of V_{n-2k} is equal to $\binom{n}{k} - \binom{n}{k-1}$ for $k = 1, \ldots, \lfloor n/2 \rfloor$, and all other irreducibles V_m have multiplicity zero.

5. Let a be a symmetric generalized Cartan matrix, i.e. a is symmetric with diagonal entries 2 and off-diagonal entries 0 or -1. Let Γ be a subgroup of the automorphism group of the

Dynkin diagram D of a, such that every edge of D has its endpoints in distinct Γ orbits. Define the *folding* D' of D to be the diagram with a node for every Γ orbit I of nodes in D, with edge weight k from I to J if each node of I is adjacent in D to k nodes of J. Denote by a' the generalized Cartan matrix with diagram D'.

(a) Show that a' is symmetrizable and that every symmetrizable generalized Cartan matrix (not assumed to be of finite type) can be obtained by folding from a symmetric one.

(b) Show that every folding of a finite type symmetric Cartan matrix is of finite type.

(c) Verify that every non-symmetric finite type Cartan matrix is obtained by folding from a unique symmetric finite type Cartan matrix.

6. An indecomposable symmetrizable generalized Cartan matrix a is said to be of *affine type* if $\det(a) = 0$ and all the proper principal minors of a are positive.

   (a) Classify the affine Cartan matrices.

   (b) Show that every non-symmetric affine Cartan matrix is a folding, as in the previous problem, of a symmetric one.

   (c) Let $\mathfrak{h}$ be a vector space, $\alpha_i \in \mathfrak{h}^*$ and $\alpha_i^\vee \in \mathfrak{h}$ vectors such that a is the matrix $\langle \alpha_j, \alpha_i^\vee \rangle$. Assume that this realization is non-degenerate in the sense that the vectors α_i are linearly independent. Define the *affine Weyl group* W to be generated by the reflections s_{α_i}, as usual. Show that W is isomorphic to the semidirect product $W_0 \ltimes Q$ where Q and W_0 are the root lattice and Weyl group of a unique finite root system, and that every such $W_0 \ltimes Q$ occurs as an affine Weyl group.

   (d) Show that the affine and finite root systems related as in (c) have the property that the affine Dynkin diagram is obtained by adding a node to the finite one, in a unique way if the finite Cartan matrix is symmetric.

7. Work out the root systems of the orthogonal Lie algebras $\mathfrak{so}(m, \mathbb{C})$ explicitly, thereby verifying that they correspond to the Dynkin diagrams B_n if $m = 2n + 1$, or D_n if $m = 2n$. Deduce the isomorphisms $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C})$, and $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$.

8. Show that the Weyl group of type B_n or C_n (they are the same because these two root systems are dual to each other) is the group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ of signed permutations, and that the Weyl group of type D_n is its subgroup of index two consisting of signed permutations with an even number of sign changes, i.e., the semidirect factor $(\mathbb{Z}/2\mathbb{Z})^n$ is replaced by the kernel of S_n-invariant summation homomorphism $(\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$

9. Let $(\mathfrak{h}, R, R^\vee)$ be a finite root system, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots with respect to a choice of positive roots R_+, $s_i = s_{\alpha_i}$ the corresponding generators of the Weyl group W. Given $w \in W$, let $l(w)$ denote the minimum length of an expression for w as a product of the generators s_i.

   (a) If $w = s_{i_1} \ldots s_{i_r}$ and $w(\alpha_j) \in R_-$, show that for some k we have $\alpha_{i_k} = s_{i_{k+1}} \ldots s_{i_r}(\alpha_j)$, and hence $s_{i_k} s_{i_{k+1}} \ldots s_{i_r} = s_{i_{k+1}} \ldots s_{i_r} s_j$. Deduce that $l(w s_j) = l(w) - 1$ if $w(\alpha_j) \in R_-$.

(b) Using the fact that the conclusion of (a) also holds for $v = ws_j$, deduce that $l(ws_j) = l(w) + 1$ if $w(\alpha_j) \notin R_-$.

(c) Conclude that $l(w) = |w(R_+) \cup R_-|$ for all $w \in W$. Characterize $l(w)$ in more explicit terms in the case of the Weyl groups of type A and B/C.

(d) Assuming that $\mathfrak{h}$ is over $\mathbb{R}$, show that the dominant cone $X = \{\lambda \in \mathfrak{h} : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\}$ is a fundamental domain for W, i.e., every vector in $\mathfrak{h}$ has a unique element of X in its W orbit.

(e) Deduce that $|W|$ is equal to the number of connected regions into which $\mathfrak{h}$ is separated by the removal of all the root hyperplanes $\langle \lambda, \alpha^\vee \rangle$, $\alpha^\vee \in R^\vee$.

10. Let $h_1, \ldots, h_r$ be linear forms in variables $x_1, \ldots, x_n$ with integer coefficients. Let $\mathbb{F}_q$ denote the finite field with $q = p^e$ elements. Prove that except in a finite number of "bad" characteristics p, the number of vectors $v \in \mathbb{F}_q^n$ such that $h_i(v) \neq 0$ for all i is given for all q by a polynomial $\chi(q)$ in q with integer coefficients, and that $(-1)^n \chi(-1)$ is equal to the number of connected regions into which $\mathbb{R}^n$ is separated by the removal of all the hyperplanes $h_i = 0$.

    Pick your favorite finite root system and verify that in the case where the h_i are the root hyperplanes, the polynomial $\chi(q)$ factors as $(q - e_1) \ldots (q - e_n)$ for some positive integers e_i called the *exponents* of the root system. In particular, verify that the sum of the exponents is the number of positive roots, and that (by Problem 9(e)) the order of the Weyl group is $\prod_i (1 + e_i)$

11. The *height* of a positive root α is the sum of the coefficients c_i in its expansion $\alpha = \sum_i c_i \alpha_i$ on the basis of simple roots.

    Pick your favorite root system and verify that for each $k \geq 1$, the number of roots of height k is equal to the number of the exponents e_i in Problem 10 for which $e_i \geq k$.

12. Pick your favorite root system and verify that if h denotes the height of the highest root plus one, then the number of roots is equal to h times the rank. This number h is called the *Coxeter number*. Verify that, moreover, the multiset of exponents (see Problem 10) is invariant with respect to the symmetry $e_i \mapsto h - e_i$.

13. A *Coxeter element* in the Weyl group W is the product of all the simple reflections, once each, in any order. Prove that a Coxeter element is unique up to conjugacy. Pick your favorite root system and verify that the order of a Coxeter element is equal to the Coxeter number (see Problem 12).

14. The *fundamental weights* λ_i are defined to be the basis of the weight lattice P dual to the basis of simple coroots in $Q^\vee$, i.e., $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$.

    (a) Prove that the stabilizer in W of λ_i is the Weyl group of the root system whose Dynkin diagram is obtained by deleting node i of the original Dynkin diagram.

    (b) Show that each of the root systems E_6, E_7, and E_8 has the property that its highest root is a fundamental weight, and identify the corresponding simple root. Deduce that the order of the Weyl group $W(E_k)$ in each case is equal to the number of roots times the order of the Weyl group $W(G)$, where G is the root system formed from E_k by deleting the identified root. Use this to calculate the orders of the Weyl groups $W(E_k)$.

15. Let $e_1, \ldots, e_8$ be the usual orthonormal basis of coordinate vectors in Euclidean space $\mathbb{R}^8$. The root system of type E_8 can be realized in $\mathbb{R}^8$ with simple roots $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, 7$ and
$$\alpha_8 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$
Show that the root lattice Q is equal to the weight lattice P, and that in this realization, Q consists of all vectors $\beta \in \mathbb{Z}^8$ such that $\sum_i \beta_i$ is even and all vectors $\beta \in \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \mathbb{Z}^8$ such that $\sum_i \beta_i$ is odd. Show that the root system consists of all vectors of squared length 2 in Q, namely, the vectors $\pm e_i \pm e_j$ for $i < j$, and all vectors with coordinates $\pm\frac{1}{2}$ and an odd number of coordinates with each sign.

16. Show that the root system of type F_4 has 24 long roots and 24 short roots, and that the roots of each length form a root system of type D_4. Show that the highest root and the highest short root are the fundamental weights at the end nodes of the diagram. Then use Problem 14(a) to calculate the order of the Weyl group $W(F_4)$. Show that $W(F_4)$ acts on the set of short (resp. long roots) as the semidirect product $S_3 \ltimes W(D_4)$, where the symmetric group S_3 on three letters acts on $W(D_4)$ as the automorphism group of its Dynkin diagram.

17. Pick your favorite root system and verify that the generating function $W(t) = \sum_{w \in W} t^{l(w)}$ is equal to $\prod_i (1 + t + \cdots + t^{e_i})$, where e_i are the exponents as in Problem 10.

18. Let S be the subring of W-invariant elements in the ring of polynomial functions on $\mathfrak{h}$. Pick your favorite root system and verify that S is a polynomial ring generated by homogeneous generators of degrees $e_i + 1$, where e_i are the exponents as in Problem 10.